GENERAL THEORY OF MODULAR INVARIANTS*

BY

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Introduction.

The discovery of the fundamental theorems, established in the first part of the present paper, on the invariants of a general system of s forms under linear transformations in a finite field was the outcome of a new standpoint for the consideration of modular invariants. In former papers on the subject (cited later), the test for the invariance of a polynomial consisted in a more or less direct verification that it remained unaltered, up to a power of the determinant of the transformation, under the general linear group G in the field; instead of certain generators of the latter, the corresponding annihilators were employed. In the present paper, the transformation concept is employed only to furnish a complete set of non-equivalent classes $C_0, \cdots, C_{f-1}$ of systems of s forms under the group G. Thus the test for the absolute invariance of a polynomial P is that P shall take the same value for all systems of s forms in a class. It is shown in § 4 that the number of linearly independent absolute invariants equals the number f of classes under the total group G. In § 6 it is shown that the number of linearly independent invariants, including both absolute and relative, equals the number of classes under the group G_1 of transformations of determinant unity; it is furthermore specified which of the invariants under G_1 are invariants of the s forms.

The general theory is applied in §§ 8, 9, 16–19 to the determination of all the invariants of the general m-ary quadratic form in the Galois field of order p^n, and in §§ 22–26 to the construction of all invariants of the binary cubic form† in the $GF[p^n]$. For the practical construction of the invariants, there is developed a uniform process, of function-theoretic nature, for the conversion of non-invariantive characterizations of the classes into invariantive characterizations. The intervening sections are devoted to the determination and characterization of the classes of the forms under investigation. A mere list of canonical types of forms is not sufficient. For m-ary quadratic forms in the $GF[2^n]$ such a list has been given‡ by the author; to obtain necessary and

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† Results for higher binary forms and for a pair of m-ary quadratic forms are to be published soon.

‡ American Journal of Mathematics, vol. 21 (1899), p. 222.

sufficient criteria for each class, a new theory for such forms has been constructed in §§ 10–15. Also for binary cubic forms, the case (§ 26) in which the modulus p equals 2 is more intricate than the general case $p > 2$. The nature of the invariants is quite different in the two cases, a result to be anticipated for quadratic forms, but rather surprising for cubic forms. The consequent assignment of such a large part of the present paper to the special case $p = 2$ was made not merely for the sake of completeness, but rather on account of the very prominent rôle which the linear groups with modulus 2 play in the applications * to geometry and in the general theory of linear groups.

Invariantive properties are often expressed in the algebraic theory by the vanishing of a covariant, or by means of an integer such as the rank of the matrix of a quadratic form. In the modular theory, every such property can be expressed by means of an explicit invariantive function of the coefficients. For quadratic forms illustrations of this point occur below. For a covariant K with the coefficients A_i,

$$I = \Pi (1 - A_i^{p^n-1})$$

is a modular invariant of the initial forms, in view of formula (1). We have $K \equiv 0$ or $K \not\equiv 0$ according as $I = 1$ or $I = 0$.

The method of the present paper for constructing modular invariants affords immediately important interpretations of these invariants.

It seems probable that modular invariants are destined to play a rôle in the theory of numbers comparable to that played by algebraic and differential invariants in higher algebra and geometry.

For a given set of forms, the theory of its modular invariants presents a doubly infinite number of problems, in view of the order p^n of the finite field. As compared with the use of annihilators employed in the earlier papers, the power of the present method may be inferred from the ease with which the various fields are now considered simultaneously. Hitherto the completeness of a proposed system of modular invariants for an infinitude of fields had not been established, even in so apparently simple a case as that of the binary quadratic form.

Existence theorems on modular invariants, §§ 1–5.

1. We shall make use of the following general theorem on interpolation in any finite field. *Within the $GF[p^n]$, there exists one and but one polynomial $\phi(x_1, \cdots, x_k)$ which has each exponent $\leqq p^n - 1$ and which takes prescribed values $v_{x_1,\ldots,x_k}$ for every set of elements $x_1, \cdots, x_k$ in the field.*

To proceed by induction, let the theorem be true for $k-1$ variables $x_2, \cdots, x_k$. Denote the elements of the field by $e_0, e_1, \cdots, e_\nu$, where $\nu = p^n - 1$. Then, for each value of i, there exists an unique polynomial $\phi(e_i, x_2, \cdots, x_k)$ with expo-

* JORDAN, *Traité des substitutions*, p. 313, p. 329; DICKSON, Annals of Mathematics, ser. II, vol. 6 (1905), pp. 141–150.

nents $\leqq \nu$, which takes the values $v_{x_1, x_2, \cdots, x_k}$ when $x_2, \cdots, x_k$ range over the field. We may now determine uniquely polynomials ψ_i in $x_2, \cdots, x_k$ such that

$$\psi_0 + \psi_1 x_1 + \cdots + \psi_\nu x_1^\nu \equiv \phi(x_1, x_2, \cdots, x_k)$$

will take the prescribed values $v_{x_1, \cdots, x_k}$. Indeed, the equations obtained by setting $x_1 = e_0, \cdots, e_\nu$ uniquely determine $\psi_0, \cdots, \psi_\nu$ as linear functions of the known polynomials $\phi(e_i, x_2, \cdots, x_k)$, since

$$\begin{vmatrix} 1 & e_0 & e_0^2 & \cdots & e_0^\nu \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & e_\nu & e_\nu^2 & \cdots & e_\nu^\nu \end{vmatrix} = \prod_{i<j} (e_i - e_j) \neq 0.$$

The argument applies also when $k = 1$, the ψ_i being parameters in the field. Hence the induction is complete.

Corollary. If two polynomials, with each exponent $\leqq p^n - 1$, are equal in $GF[p^n]$ for all sets of values of the variables, they are identical.

2. Consider a system of forms $F_1, \cdots, F_s$, where F_i is the general polynomial of degree d_i in m variables having as coefficients arbitrary parameters in the $GF[p^n]$. Assigning particular values to these parameters, we obtain a particular system S_1 of s forms. The distinct systems that can be obtained from S_1 by applying the various transformations of a given m-ary linear homogeneous group* L in the $GF[p^n]$ constitute the *class* C_1 of systems of forms conjugate, under L, with the given system S_1. Since the field is of finite order p^n, the order of L is finite, the number of systems in one class is finite, and there is a finite number of classes $C_0, C_1, \cdots, C_{f-1}$.

Selecting arbitrarily a system S_i from the class C_i, we shall say that $S_0, \cdots, S_{f-1}$ constitute a complete set of non-equivalent systems, under L, of s forms of degrees $d_1, \cdots, d_s$. When the S_i are relatively simple representatives of their classes, they are said to form a complete set of canonical types under L.

We shall expressly include the class, henceforth designated by C_0, which is composed of the system of forms all of whose coefficients are zero (see end of § 4).

3. THEOREM. *Under a given linear homogeneous group L in the $GF[p^n]$, a system of forms has one and but one invariant† which takes prescribed values $v_0, \cdots, v_{f-1}$ for the various classes $C_0, \cdots, C_{f-1}$.*

By § 1, there exists one and but one polynomial P which takes the prescribed values. If any chosen transformation of L replaces P by P', then P' takes the same values, so that $P' \equiv P$.

* The replacement of a group by any set of transformations is only an apparent generalization.

† A polynomial in the coefficients a_j of the forms, each a entering to a power $\leqq p^n - 1$, as may be assumed in view of $a^{p^n} = a$.

4. Denote by I_k the uniquely determined invariant which has the value unity for the class C_k and the value zero for every class $C_i (i \neq k)$. Since I_k completely characterizes the class C_k, it will be called the *characteristic invariant of the system of forms for the class C_k under the group L.*

Let $\sum c_i I_i \equiv 0$ be a linear homogeneous relation between the invariants I_i, with constant coefficients c_i in the $GF[p^n]$. Assign to the coefficients a_j of the given system of forms the values which they have in a particular system of forms belonging to the class C_k. Then $I_k = 1$, $I_i = 0$ $(i \neq k)$. Hence $c_k = 0$. Thus *the characteristic invariants are linearly independent in the field.*

Consider any invariant I of the system of forms under the group L. Let I have the value v_i for the class C_i. Then, by the corollary in § 1, $I \equiv \sum v_i I_i$. *Any invariant is a linear homogeneous function of the characteristic invariants with constant coefficients in the field.*

The essential part of the preceding results is contained in the

THEOREM. *The number of linearly independent invariants of a given system of s forms under a given linear homogeneous group L in the $GF[p^n]$ equals the number of non-equivalent classes under L.*

We note the non-homogeneous relation $\sum I_i = 1$. To compensate for the inclusion of the trivial invariant unity in counting the number of linearly independent invariants, we note that we have also included the trivial class C_0 of identically vanishing forms. For the latter,

$$(1) \qquad I_0 = \prod_{i=1}^{\sigma} (1 - a_i^{p^n - 1}),$$

where $a_1, \cdots, a_\sigma$ is the aggregate of the coefficients of the s forms.

5. THEOREM. *Any set of $f - 1$ of the f characteristic invariants are independent, in the sense that no invariant of such a set equals a rational integral function of the remaining invariants of that set.*

Let the given set contain the $I_j (j \neq g)$, and assume that I_k equals a polynomial P in the $I_j (j \neq g, k)$. Since $I_i^2 = I_i$, $I_i I_j = 0 (i \neq j)$, $I_k = P$ may be given the form $I_k = c + \sum c_j I_j$ $(j \neq g, k)$. Replace c by $c \sum I_i$. The resulting homogeneous relation must be an identity. But, by the coefficients of I_k and I_g, $c = 1$, $c = 0$, respectively.

To give a second proof, consider representatives S_k and S_g of the classes C_k and C_g $(k \neq g)$. For the particular values of the coefficients in S_k and those in S_g, the invariant I_k takes different values (1 and 0), whereas each invariant $I_i (i \neq k, i \neq g)$ takes the same value zero. Hence I_k cannot equal a polynomial in the $I_i (i \neq k, g)$.

Absolute and relative invariants of a system of forms.

6. When L is the group G of all m-ary linear homogeneous transformations in the $GF[p^n]$, the invariants defined in §§ 3, 4 are called the *absolute invariants* of the s forms. When L is the group G_1 of all transformations of determinant unity, those invariants of G_1 which are multiplied by Δ^w under every transformation of determinant Δ are called the *relative invariants of weight w* of the s forms. A knowledge of the classes of forms under G_1 will be shown to be sufficient for the construction of all relative invariants.

Each class C_i under G separates into certain classes $C_{i1}, C_{i2}, \cdots, C_{ig_i}$ under G_1, which are transformed amongst themselves by all transformations of the p^n-1 possible determinants. Hence $e_i=(p^n-1)/g_i$ is the number of distinct determinants of the automorphs of a class C_{ij}. These e_i distinct values may be expressed as powers $\rho^\mu, \rho^{\mu'}, \cdots$ of a primitive root ρ of the field, where each exponent is $\geqq 0$. Let μ be the minimum positive exponent. Then $\mu'=k\mu+\nu(0\leqq\nu<\mu)$. Let T and M be automorphs of determinants $\rho^{\mu'}$ and ρ^μ, respectively. The determinant of TM^{-k} is ρ^ν. Hence $\nu=0$, so that every exponent is a multiple of μ. There are $(p^n-1)/\mu$ distinct powers of ρ^μ. Hence $\mu=g_i$ and the e_i distinct determinants of the automorphs of C_{ij} are the distinct powers of a primitive root ρ^{g_i} of $x^{e_i}=1$.

Under a linear transformation R of determinant ρ, the classes $C_{i1}, \cdots, C_{ig_i}$ undergo a permutation P. Since the determinant of R^{g_i} is a root of $x^{e_i}=1$, P^{g_i} is the identity. Since ρ^{g_i} is the least positive power of ρ which gives a root of $x^{e_i}=1$, P^{g_i} is the least positive power of P which leaves a class C_{ij} unaltered. Hence P permutes $C_{i1}, \cdots, C_{ig_i}$ in a single cycle. By assigning a suitable sequence to the $C_{ij}(j=1, \cdots, g_i)$, we may set

$$P=\prod_i(C_{i1}C_{i2}\cdots C_{ig_i}).$$

Let a_t be the general coefficient of the given system of forms, a'_t the corresponding coefficient of the forms obtained by applying the transformation R. Thus R transforms a function $V(a)$ of the coefficients a_t into $V(a')=V'(a)$. Let V be an invariant under the group G_1, so that $V(a)$ has the same value for all systems of forms in a class C_{ij}. Now R transforms the class C_{ij} into the class C_{ij+1}, which is to be identified with C_{i1} when $j=g_i$. Hence the value of V for the class C_{ij+1} equals the value of the transformed function V' for the class C_{ij}. Let V be the characteristic invariant I_{ij+1}, so that I_{ij+1} has the value unity for the class C_{ij+1} and the value zero for all the remaining classes under G_1. Hence I'_{ij+1} equals unity for C_{ij} and zero for the remaining classes. Thus I'_{ij+1} equals the characteristic invariant I_{ij} for the class C_{ij}. Hence

$$R\sim\prod_i(I_{ig_i}\cdots I_{ij+1}I_{ij}\cdots I_{i2}I_{i1}).$$

Now $\gamma_i = \rho^{e_i}$ belongs to the exponent g_i. The g_i functions

$$\Sigma_{ik} = I_{i1} + \gamma_i^k I_{i2} + \gamma_i^{2k} I_{i3} + \cdots + \gamma_i^{(g_i-1)k} I_{ig_i} \quad (k = 0, 1, \cdots, g_i - 1) \tag{2}$$

are linearly independent functions of $I_{i1}, \cdots, I_{ig_i}$. The

$$g = g_0 + g_1 + \cdots + g_{f-1}$$

characteristic invariants $I_{ij} (j = 1, \cdots, g_i;\ i = 0, 1, \cdots, f-1)$ are linearly independent (§ 4). Hence the g invariants Σ_{ik} are linearly independent. Now R replaces Σ_{ik} by $\gamma_i^k \Sigma_{ik}$. Thus R^t, of determinant ρ^t, multiplies Σ_{ik} by

$$\gamma_i^{kt} = (\rho^t)^{ke_i}.$$

Hence Σ_{ik} is a relative invariant of weight ke_i.

Let $K = \sum c_{ij} I_{ij}$ be any relative invariant of weight w. Under the transformation R, of determinant ρ, K becomes $\rho^w K$, while I_{ij} becomes I_{ij-1} or I_{ig_i} according as $j > 1$ or $j = 1$. Hence for every $i, j\ (j > 1)$,

$$c_{ij} = \rho^w c_{ij-1}, \qquad c_{i1} = \rho^w c_{ig_i}.$$

Unless every $c_{ij} = 0$, we must have $\rho^{wg_i} = 1$, so that wg_i is a multiple of $p^n - 1 = e_i g_i$, whence $w = ke_i$. In the latter case, $\rho^w = \gamma_i^k$,

$$\sum_{j=1}^{g_i} c_{ij} I_{ij} = c_{i1} \sum_{j=1}^{g_i} \gamma_i^{k(j-1)} I_{ij} = c_{i1} \Sigma_{ik}.$$

Hence K is a linear function of those Σ_{ik} which are of weight w.

We have now established the following

THEOREM. *For a system of s general forms of given degrees in m variables, with arbitrary coefficients in the $GF[p^n]$, the number of linearly independent invariants, absolute and relative, equals the number g of the classes of the systems of forms under the group G_1 of m-ary linear transformations of determinant unity.*

Of the f classes C_i under the total m-ary linear group G, let the class C_i separate into g_i classes C_{ij} under G_1. Let k be any integer such that $0 \leqq k < g_i$. Then, for each i, there exist g_i invariants Σ_{ik}, given by (2), of weights ke_i, where $e_i = (p^n - 1)/g_i$. The $g = g_0 + \cdots + g_{f-1}$ invariants Σ_{ik} are linearly independent. Any invariant of weight w of G is a linear homogeneous function of those of the invariants Σ_{ik} which are of weight w.*

* By introducing notations to indicate which of the numbers e_i are equal, we may readily give an explicit formula for the total number of invariants, as well as the number of relative invariants of each weight. For example, the number of absolute invariants is p^{nf}.

The classes of quadratic forms q_m in the $GF[p^n]$, $p>2$.

7. First, let the field be the $GF[p^n]$, $p>2$, and consider the form

$$q_m=\sum_{i,j=1}^{m}\beta_{ij}x_ix_j \qquad (\beta_{ji}=\beta_{ij}), \tag{3}$$

whose coefficients are arbitrary parameters in the field. Then

$$D=|\beta_{ij}| \tag{3'}$$

is called the determinant or discriminant of q_m. For particular values in the field for the coefficients, q_m is said to be of rank zero if every $\beta_{ij}=0$, and of rank $r>0$ if at least one minor of order r of D is not zero, while every minor of order greater than r vanishes.

Of various methods leading to a classification of quadratic forms that due to KRONECKER * is best suited for the present application; his theory is seen to hold for any field not having modulus 2. It is based on two theorems. First, in a symmetrical matrix (β_{ij}) of rank $r>0$, not every principal minor of order r vanishes. Next, if the principal minor

$$D_{k_1,\cdots,k_r}\equiv|\beta_{k_sk_t}| \qquad (s,t=1,\cdots,r) \tag{4}$$

is not zero, while every minor of order $>r$ vanishes, there exists a linear transformation of determinant unity which replaces q_m by

$$\sum_{s,t=1}^{r}\beta_{k_sk_t}x_{k_s}x_{k_t}. \tag{5}$$

A proof of the last statement will be given in a form convenient for comparison with the treatment in § 14 of the case in which the modulus is 2. After rearranging the variables, we may assume that, instead of (4),

$$|\beta_{ij}|\neq 0 \qquad (i,j=1,\cdots,r). \tag{4'}$$

To q_m we apply the transformation, of determinant unity,

$$x_i=x_i'+c_ix_m' \ (i\leqq r), \qquad x_i=x_i' \ (i>r),$$

and obtain the form

$$\sum_{i,j=1}^{m-1}\beta_{ij}x_i'x_j'+2\sum_{j=1}^{m-1}B_{jm}x_j'x_m'+Ax_m'^2,$$

$$B_{jm}\equiv\sum_{i=1}^{r}\beta_{ij}c_i+\beta_{jm}, \qquad A\equiv\sum_{j=1}^{r}B_{jm}c_j+B_{mm}.$$

In view of (4'), $c_1,\cdots,c_r$ can be uniquely determined so that $B_{jm}=0$ $(j\leqq r)$. In the latter and B_{km} $(r<k\leqq m)$, the determinant of the coefficients of $c_1,\cdots,c_r,1$ is the minor of β_{km} in

$$|\beta_{ij}| \qquad (i,j=1,\cdots,r,k,m).$$

* *Werke*, vol. 1, p. 166, p. 357. Cf. BÔCHER, *Introduction to Higher Algebra*, p. 58, p. 139; GUNDELFINGER, Journal für reine u. angewandte Mathematik, vol. 91 (1881), p. 221.

This minor of order $r+1$ is zero by hypothesis. Hence $B_{km}=0\,(k>r)$, $A=0$.

Repetitions of the process lead to the form

$$\sum_{i,j=1}^{r}\beta_{ij}x_ix_j. \tag{5'}$$

By a linear transformation of determinant unity, form (5) or (5'), of non-vanishing determinant ρ, can be transformed into *

$$\sum_{i=1}^{r-1}x_i^2+\rho x_r^2; \tag{6}$$

while a linear transformation of determinant Δ replaces (5') by a form of determinant $\Delta^2\rho$. Hence two forms (6_1) and (6_2) are equivalent under the group G_1 of all r-ary linear transformations of determinant unity only when $\rho_1=\rho_2$; but are equivalent under the total group G if, and only if, ρ_1/ρ_2 is a square in the field.

Under the total group G, the class to which the form (6), with $\rho\neq 0$, belongs will be designated by $C_{r,1}$ or $C_{r,-1}$ according as ρ is a square or a not-square. Thus a complete set of non-equivalent classes of m-ary quadratic forms in the $GF[p^n]$, $p>2$, is given by

$$C_0,\ C_{r,\pm 1} \qquad (r=1,\cdots,m). \tag{7}$$

Hence a particular form q_m of rank $r>0$ belongs to the class $C_{r,\delta}$, where $\delta=D^{\mu}_{k_1,\ldots,k_r}$, $D_{k_1,\ldots,k_r}$ being a non-vanishing principal minor of order r of $|q_m|$, and $\mu=\frac{1}{2}(p^n-1)$.

The invariants of a quadratic form in the $GF[p^n]$, $p>2$.

8. We proceed to construct the characteristic absolute invariants I_0, $I_{r,\pm 1}$ of the general quadratic form q_m in the $GF[p^n]$, $p>2$, which correspond to the classes (7). By (1),

$$I_0=\Pi(1-\beta_{ij}^{2\mu}) \qquad (i,j=1,\cdots,m;\ i\leqq j). \tag{8}$$

Instead of dealing initially with the $I_{r,\pm 1}$, we first construct an absolute invariant A_r which takes the value $+1$ for the class $C_{r,+1}$, the value -1 for the class $C_{r,-1}$, and the value 0 for the remaining classes C_0, $C_{s,\pm 1}\,(s\neq r)$. Then

$$I_{r,+1}=\tfrac{1}{2}(A_r^2+A_r), \qquad I_{r,-1}=\tfrac{1}{2}(A_r^2-A_r). \tag{9}$$

For $r=m$, we evidently have

$$A_m=D^{\mu} \qquad [\mu=\tfrac{1}{2}(p^n-1)]. \tag{10}$$

For $0<r<m$, the results of § 7 show (*a*) that $A_r=0$ if any principal minor of order $>r$ is not zero; (*b*) that $A_r=0$ if every principal minor of order $\geqq r$ vanishes; and (*c*) that $A_r=M^{\mu}$ if a particular principal minor M of order r is not zero, while every principal minor of order $>r$ vanishes. In view of the

* DICKSON, *Linear Groups*, pp. 157–158.

first statement (a), we have the identity

$$A_r \equiv \alpha_r \Pi (1 - d^{2\mu}), \tag{11}$$

where d ranges over the principal minors of order $> r$, while α_r is a polynomial in the β_{ij} to be determined. Let $M_1, \cdots, M_\rho$ denote the $\rho = C_r^m$ principal minors of order r, arranged in some definite sequence. In view of the third statement (c),

$$\alpha_r = M_i^\mu \qquad (\text{for every } d = 0,\ M_i \neq 0). \tag{12}$$

For $i = 1$, the latter yields the identity

$$\alpha_r \equiv M_1^\mu + K_1(1 - M_1^{2\mu}) \qquad (\text{for every } d = 0). \tag{13}$$

To determine K_1, consider any set of values of the β_{ij} which make each $d = 0$, $M_1 = 0$, $M_2 \neq 0$. Then by (12) for $i = 2$, and (13), $M_2^\mu = K_1$. Thus

$$K_1 \equiv M_2^\mu + K_2(1 - M_2^{2\mu}) \qquad (\text{for every } d = 0,\ M_1 = 0).$$

Then (13) yields* (14) for the case $j = 2$:

$$\begin{aligned} \alpha_r \equiv\ & M_1^\mu + M_2^\mu(1 - M_1^{2\mu}) + M_3^\mu(1 - M_1^{2\mu})(1 - M_2^{2\mu}) + \cdots \\ & + M_j^\mu(1 - M_1^{2\mu}) \cdots (1 - M_{j-1}^{2\mu}) + K_j(1 - M_1^{2\mu}) \cdots (1 - M_j^{2\mu}), \end{aligned} \tag{14}$$

for every set β_{ij} such that each $d = 0$. For the general step in the derivation of (14), we proceed by induction from $j = t$ to $j = t + 1$, assuming that (14) holds for $j = t$. We consider any set β_{ij} for which each $d = 0$, $M_i = 0\,(i \leqq t)$, $M_{t+1} \neq 0$. Then by (12) for $i = t + 1$, and (14) for $j = t$, $M_{t+1}^\mu = K_t$. Thus

$$K_t \equiv M_{t+1}^\mu + K_{t+1}(1 - M_{t+1}^{2\mu}),$$

for all sets β_{ij} such that each $d = 0$, $M_i = 0\,(i \leqq t)$. Upon substituting this value of K_t in (14), for $j = t$, we obtain (14), for $j = t + 1$.

Having established (14) for every $j \leqq \rho$, where ρ is the total number of the minors M, we consider (14) for $j = \rho$. Let the β_{ij} have any values such that $M_i = 0\,(i \leqq \rho)$ and every $d = 0$. By the second statement (b) preceding (11), $A_r = 0$. Thus, by (14) for $j = \rho$, $0 = K_\rho$. Independently of the restriction that every $d = 0$, (11) and (14) now lead to the result that *the explicit expression for the absolute invariant A_r is*

$$A_r = \{M_1^\mu + M_2^\mu(1 - M_1^{2\mu}) + \cdots + M_\rho^\mu(1 - M_1^{2\mu}) \cdots (1 - M_{\rho-1}^{2\mu})\} \Pi(1 - d^{2\mu}), \tag{15}$$

where d ranges over the principal minors of orders $> r$, while $M_1, \cdots, M_\rho$ denote the principal minors of order r taken in any sequence.

The range of d may be restricted to the principal minors of orders $r + 1$, $r + 2$.

* The restriction $M_1 = 0$ or K_1 is suppressed in (14) in view of $(1 - M_1^{2\mu})\, M_1 = 0$.

We note several forms of invariant (15) when $m=2$, $r=1$. In that case,

$$(16)\quad A_1=(\beta_{22}^{\mu}+\beta_{11}^{\mu}-\beta_{22}^{2\mu}\beta_{11}^{\mu})(1-D^{2\mu})=(\beta_{11}^{\mu}+\beta_{22}^{\mu}-\beta_{11}^{2\mu}\beta_{22}^{\mu})(1-D^{2\mu}),$$

$$(17)\qquad A_1=\tfrac{1}{2}(2-\beta_{11}^{\mu}\beta_{22}^{\mu})(\beta_{11}^{\mu}+\beta_{22}^{\mu})(1-D^{2\mu}).$$

Applying the useful congruence

$$(18)\quad (a-b)^{2\mu}\equiv\sum_{i=0}^{2\mu}a^i b^{2\mu-i}=(a^{\mu}+b^{\mu})\sum_{i=0}^{\mu}a^i b^{\mu-i}-a^{\mu}b^{\mu}\pmod p,$$

for $a=\beta_{11}\beta_{22}$, $b=\beta_{12}^2$, we get

$$\begin{aligned}D^{2\mu}-1&=(\beta_{11}^{\mu}\beta_{22}^{\mu}+1+\beta_{12}^{2\mu}-1)\sum_{i=1}^{\mu}\beta_{11}^{i}\beta_{22}^{i}\beta_{12}^{2\mu-2i}-\beta_{11}^{\mu}\beta_{22}^{\mu}\beta_{12}^{2\mu}-1\\&=(\beta_{11}^{\mu}\beta_{22}^{\mu}+1)(-1+\Sigma),\end{aligned}$$

since $(\beta_{12}^{2\mu}-1)\Sigma=(\beta_{12}^{2\mu}-1)\beta_{11}^{\mu}\beta_{22}^{\mu}$. Hence (16_1) gives*

$$(19)\qquad A_1=(\beta_{11}^{\mu}+\beta_{22}^{\mu})\left(1-\sum_{i=0}^{\mu}\beta_{11}^{i}\beta_{22}^{i}\beta_{12}^{2\mu-2i}\right).$$

9. Under the group G_1 of all m-ary linear transformations of determinant unity in the $GF[p^n]$, $p>2$, a complete set of non-equivalent classes of m-ary quadratic forms is given by

$$(20)\qquad C_0,\ C_{r,\pm 1}(r=1,\cdots,m-1),\ C_{m,\rho}\ (\rho\text{ arbitrary}\neq 0),$$

where $C_{m,\rho}$ is the class containing (6) for $r=m$. In that case, different ρ's give non-equivalent forms under G_1 (§ 7). But for $r<m$, (6) is transformed into a like form, with the parameter ρa^2, by the transformation

$$x_r=ax_r',\qquad x_m=a^{-1}x_m',\qquad x_i=x_i'\qquad (i\neq r,m)$$

of determinant unity. Let ϕ be a relative invariant of weight w of q_m, so that ϕ becomes $\Delta^w\phi$ under a transformation of determinant Δ. But each class is transformed into itself by every transformation of determinant ± 1 (in particular, by the one changing the sign of x_1). Hence w is even (cf. § 6).

Of the classes (7) under the total group G, classes $C_{m,\pm 1}$ alone separate into subclasses $C_{m,\rho}$ under G_1. If ϕ has the value v for a class $C_{m,\rho}$, then ϕ has the value $\Delta^w v$ for the class $C_{m,\rho\Delta^2}$ obtained from $C_{m,\rho}$ by a transformation of determinant Δ. Hence, if ρ'/ρ is a square, the value of ϕ for $C_{m,\rho'}$ equals $(\rho'/\rho)^{w/2}$ times its value for $C_{m,\rho}$. Thus arbitrary values can be assigned to ϕ for just one of the $\mu=\frac{1}{2}(p^n-1)$ classes $C_{m,\rho}$ in which ρ is a square, and for just one of the μ classes $C_{m,\rho}$ in which ρ is a not-square. When these two assigned values are zero, the invariant is absolute and has been constructed in § 8. The

* Invariant (19) is the negative of Q, these Transactions, vol. 8 (1907), pp. 211, 217, 218.

same is true if a non-vanishing value be assigned to ϕ for one of the classes C_0, $C_{r,\pm 1}$ $(r<m)$. Hence to obtain a non-absolute invariant ϕ of weight 2δ, where $0<2\delta<p^n-1$, we must assign to ϕ the value zero for C_0, $C_{r,\pm 1}$ $(r<m)$, and values not both zero for two classes $C_{m,\rho}$, in one of which ρ is a particular square, in the other ρ is a particular not-square. Since the values of ϕ for every class are then uniquely determined, an unique polynomial ϕ can be constructed (§ 1). Hence there are exactly two linearly independent invariants of weight 2δ. But, if D is the determinant of q_m, then D^δ and $D^{\delta+\mu}$ are linearly independent invariants of weight 2δ.

Combining the present results with those in § 8, we obtain the

THEOREM. *A complete set of linearly independent invariants of the m-ary quadratic form in the $GF[p^n]$, $p>2$, is given by the $2m+1$ characteristic absolute invariants I_0, $I_{r,\pm 1}$ $(r=1,\cdots,m)$, and the p^n-3 relative invariants D^δ, $D^{\delta+\mu}$ $(\delta=1,\cdots,\mu-1)$, where $\mu=\frac{1}{2}(p^n-1)$. As an alternative set, we may take*

$$(21)\qquad I_0,\ A_r,\ A_r^2\ (r=1,\cdots,m-1),\qquad D^k\ (k=1,\cdots,p^n-1).$$

The invariants mentioned are defined by (3′), (8), (9), (10), (15).

The product of any two invariants of the first set can be reduced to a linear combination of those in the first set by means of the relations*

$$(22)\qquad \begin{gathered} D^{2\mu+1}=D,\qquad I_0D=I_{r,\pm 1}D=0\,(r<m),\qquad I_{m,\pm 1}D=\tfrac{1}{2}(D\pm D^{\mu+1}),\\ I^2=I,\qquad II'=0\,(I \text{ and } I' \text{ any pair of the } I_0,\ I_{r,\pm 1}). \end{gathered}$$

For the set (21), the following relations suffice:

$$(23)\quad D^{2\mu+1}=D,\ I_0^2=I_0,\ A_r^3=A_r,\ I_0A_r=I_0D=A_rD=A_rA_s=0\,(r\neq s).$$

Reduction of quadratic forms in the $GF[2^n]$, §§ 10–15.

10. In view of the application to be made for fields having the modulus 2, we consider some algebraic properties of the quadratic form

$$(24)\qquad Q_m=\Sigma\beta_{ij}x_ix_j+\Sigma b_jx_j^2\qquad (i,j=1,\cdots,m\,;\ i<j).$$

Its discriminant D and a related skew-symmetric determinant d are

$$D=\begin{vmatrix} 2b_1 & \beta_{12} & \cdots & \beta_{1m}\\ \beta_{12} & 2b_2 & \cdots & \beta_{2m}\\ \cdot & \cdot & \cdots & \cdot\\ \beta_{1m} & \beta_{2m} & \cdots & 2b_m \end{vmatrix},\qquad d=\begin{vmatrix} 0 & \beta_{12} & \cdots & \beta_{1m}\\ -\beta_{12} & 0 & \cdots & \beta_{2m}\\ \cdot & \cdot & \cdots & \cdot\\ -\beta_{1m} & -\beta_{2m} & \cdots & 0 \end{vmatrix}.$$

Now $d\equiv 0$ for m odd; while, for m even, d equals the square of the pfaffian †

* These follow at once from the values of the invariants for the various classes.

† We consider pfaffians in the β_{ij} exclusively. Note that [12] denotes β_{12}.

$[12\cdots m]$. Since corresponding elements of D and d differ by multiples of 2,

$$(25)\qquad D = 2S_m \quad (m \text{ odd}), \qquad D = [12\cdots m]^2 + 2R \quad (m \text{ even}).$$

If Q_m becomes Q'_m under a linear homogeneous transformation of determinant Δ, the discriminant D' of Q'_m equals $\Delta^2 D$. Hence, for m odd, S_m is invariant up to the factor Δ^2; while, for m even,

$$[12\cdots m]_{\beta'} = \Delta\,[12\cdots m]_\beta + 2\rho,$$

since the β'_{ij} involve only even multiples of the b_k.

11. Let the coefficients of Q_m belong to the $GF[2^n]$. If any coefficient be increased by a multiple of 2, D is increased by a multiple of 4; thus S_m and $[12\cdots m]$ are increased by multiples of 2. It will prove convenient to employ the notation $\{12\cdots m\}$ for S_m. *According as m is even or odd,* $[12\cdots m]$ *or* $\{12\cdots m\}$ *is an invariant* of* (24) *in the* $GF[2^n]$.

We shall consider two methods of normalizing Q_m under linear transformation in the $GF[2^n]$, each possessing certain advantages. The contrast between the two methods is analogous to that between the methods of LAGRANGE and KRONECKER in the algebraic theory. The present theory is essentially different from the algebraic theory; this is due partly to the fact that the terms involving the squares of the variables hold themselves aloof under transformation.

12. If every $\beta_{ij} = 0$, Q_m is the square of $\sum b_j^{\frac{1}{2}} x_j$ in the $GF[2^n]$ and either vanishes identically or can be transformed into x_1^2. If not every β_{ij} vanishes, let $\beta_{12} \neq 0$, $m > 2$. Under the transformation

$$(26)\quad x'_1 = x_1 + \sum_{i=3}^{m} \beta_{2i} x_i, \qquad x'_2 = x_2 + \sum_{i=3}^{m} \beta_{1i} x_i, \qquad x'_j = \beta_{12} x_j \quad (j = 3, \cdots, m),$$

of determinant β_{12}^{m-2}, $Q_m(x')$ becomes †

$$(27)\quad \beta_{12} x_1 x_2 + \beta_{12} \sum [12ij] x_i x_j + b_1 x_1^2 + b_2 x_2^2 + \sum \{12i\} x_i^2 \quad (i, j = 3, \cdots, m;\ i < j),$$

where $[12ij]$ is a pfaffian, while

$$\{12i\} = \beta_{12}\beta_{1i}\beta_{2i} + b_1 \beta_{2i}^2 + b_2 \beta_{1i}^2 + b_i \beta_{12}^2,$$

so that $\{123\}$ is the semi-discriminant of Q_3. Hence, for $m = 3$ or 4, the vanishing of the invariant $\{123\}$ or $\{1234\}$ is a necessary and sufficient condition that Q_3 or Q_4, with $\beta_{12} \neq 0$, shall be tranformable into a binary or ternary form, respectively.

Next, let $m > 4$. If every $[12ij] = 0$, (27) can be transformed into a ternary form. In the contrary case, let $[1234] \neq 0$ and apply the following discussion for $l = 2$, with $[1\cdots 0]$ replaced by unity, $\{1\cdots 0k\}$ replaced by b_k.

* American Journal of Mathematics, vol. 30 (1908), p. 265.

† Ibid., p. 264, formula (2), with x_2 replaced by $c_{12}x_2$.

The general step in the reduction process will be made by induction. Let l be an integer $\leqq m/2$ such that not every pfaffian of order $2l$ vanishes; in particular, let

$$[12] \neq 0,\ [1234] \neq 0,\ \cdots,\ [1\cdots 2l] \neq 0. \tag{28}$$

We assume that, after $l-1$ steps, Q_m has been transformed into

$$\begin{aligned}&[12]x_1x_2+[12][1234]x_3x_4+[1234][1\cdots 6]x_5x_6+\cdots\\&+[1\cdots 2l-4][1\cdots 2l-2]x_{2l-3}x_{2l-2}+[1\cdots 2l-2]\textstyle\sum[1\cdots 2l-2\,ij]x_ix_j\\&+b_1x_1^2+b_2x_2^2+\{123\}x_3^2+\{124\}x_4^2+\cdots+\{1\cdots 2l-4\,2l-3\}x_{2l-3}^2\\&+\{1\cdots 2l-4\,2l-2\}x_{2l-2}^2+\textstyle\sum\{1\cdots 2l-2\,i\}x_i^2\quad(i,j=2l-1,\cdots,m;\ i<j),\end{aligned} \tag{29}$$

by a transformation of determinant

$$[12]^2[1234]^2\cdots[1\cdots 2l-4]^2[1\cdots 2l-2]^{m-2l+2}. \tag{30}$$

To (29), with each x accented, we apply the transformation

$$\begin{aligned}x'_{2l-1}=x_{2l-1}+t\sum_{i=2l+1}^{m}[1\cdots 2l-2\,2li]x_i,\quad x'_{2l}=x_{2l}+t\sum_{i=2l+1}^{m}[1\cdots 2l-2\,2l-1i]x_i,\\ x'_j=t[1\cdots 2l]\,x_j\qquad(j=2l+1,\cdots,m),\end{aligned} \tag{31}$$

where $t=[1\cdots 2l-2]^{-1}$. This alters only the terms of (29) under the two summation signs. These terms are replaced by

$$\begin{aligned}&[1\cdots 2l-2][1\cdots 2l]x_{2l-1}x_{2l}+[1\cdots 2l]\textstyle\sum\sigma_{ij}x_ix_j\\&+\{1\cdots 2l-2\,2l-1\}x_{2l-1}^2+\{1\cdots 2l-2\,2l\}x_{2l}^2+\textstyle\sum f_ix_i^2\\&\qquad(i,j=2l+1,\cdots,m;\ i<j),\end{aligned}$$

where

$$\begin{aligned}\sigma_{ij}=t\{&[1\cdots 2l-2\,2li][1\cdots 2l-2\,2l-1j]\\&+[1\cdots 2l-2\,2lj][1\cdots 2l-2\,2l-1i]+[1\cdots 2l][1\cdots 2l-2\,ij]\},\end{aligned} \tag{32}$$

$$\begin{aligned}f_i=t^2\big(&[1\cdots 2l-2\,2li]^2\{1\cdots 2l-2\,2l-1\}\\&+[1\cdots 2l-2\,2l-1i]^2\{1\cdots 2l-2\,2l\}+[1\cdots 2l]^2\{1\cdots 2l-2\,i\}\\&+[1\cdots 2l-2][1\cdots 2l][1\cdots 2l-2\,2l-1i][1\cdots 2l-2\,2li]\big).\end{aligned} \tag{33}$$

Hence the induction will be complete if it is shown that

$$\sigma_{ij}=[1\cdots 2l\,ij],\qquad f_i=\{1\cdots 2l\,i\}. \tag{34}$$

The product of the determinant of (31) by (30) is clearly

$$[12]^2[1234]^2\cdots[1\cdots 2l-2]^2[1\cdots 2l]^{m-2l}. \tag{35}$$

Although (34_1) may be established by means of the algebraic theory of

pfaffians, we shall prove both parts of (34) by applying the invariance of the discriminant or semi-discriminant, up to a factor Δ^2, under a transformation of determinant Δ in the $GF[2^n]$. We note that the preceding development is valid for any $m \geqq 2l$. First, we take $m = 2l+2$. Then the discriminant of the final quadratic form is seen to equal $B^2\sigma^2_{2l+1\,2l+2}$, where B is the product (35), the final exponent being 2. But the discriminant of Q_{2l+2} is $[1\cdots 2l+2]^2$. Hence (34_1) is true for $i = 2l+1$, $j = 2l+2$, and therefore for any i, j exceeding $2l$. Next for $m = 2l+1$, the semi-discriminant of the final form equals $P^2 f_{2l+1}$, where P is the product (35) with the last exponent unity. But the semi-discriminant of Q_{2l+1} is $\{1\cdots 2l+1\}$. Hence (34_2) holds for $i = 2l+1$, and therefore for every $i > 2l$.

We make the further assumption that every pfaffian of order $> 2l$ vanishes. The form reached above thus becomes

$$(36)\qquad \sum_{s=1}^{l}[1\cdots 2s-2][1\cdots 2s]x_{2s-1}x_{2s}+\sum_{i=2l+1}^{m}\{1\cdots 2l\,i\}x_i^2 + \sum_{s=1}^{l}\{1\cdots 2s-2\;2s-1\}x_{2s-1}^2+\sum_{s=1}^{l}\{1\cdots 2s-2\;2s\}x_{2s}^2.$$

To this form we apply the transformation

$$x_{2s}=[1\cdots 2s]^{-1}x'_{2s},\quad x_{2s-1}=[1\cdots 2s-2]^{-1}x'_{2s-1}\,(s\leqq l),\quad x_i=x'_i\,(i>2l),$$

and then drop the accents on the x'. We get

$$(37)\qquad \sum_{s=1}^{l}x_{2s-1}x_{2s}+\sum_{i=1}^{2l}\delta_i x_i^2+\sum_{i=2l+1}^{m}\{1\cdots 2l\,i\}x_i^2,$$

$$(38)\qquad \begin{aligned}\delta_{2s-1}&=[1\cdots 2s-2]^{-2}\{1\cdots 2s-2\;2s-1\},\\ \delta_{2s}&=[1\cdots 2s]^{-2}\{1\cdots 2s-2\;2s\}\end{aligned}\qquad (s=1,\cdots,l).$$

The product of the determinant of this transformation by (35) is

$$(39)\qquad [1\cdots 2l]^{m-2l-1}.$$

Hence Q_m is transformed into (37) by a transformation of determinant (39).

First, let at least one of the $\{1\cdots 2l\,i\}$ be not zero, where $i > 2l$, thus implying that $m > 2l$. Then, by (37), Q_m can be transformed into

$$(40)\qquad \sum_{s=1}^{l}x_{2s-1}x_{2s}+cx_{2l+1}^2\quad (c=1 \text{ if } m>2l+1,\ c=\{1\cdots 2l+1\} \text{ if } m=2l+1),$$

by a transformation of determinant unity. Under the group of transformations of all determinants, (40) with $m = 2l+1$ is equivalent to a like form with $c = 1$. In view of the invariance of the semi-discriminant, (40) can not be transformed into a form on fewer than $2l+1$ variables. Now $\{1\cdots 2l\,i\} \neq 0$

implies, in view of (33) and (34), that not every pfaffian of order $2l$ is zero.

Hence necessary and sufficient conditions that Q_m shall be transformable into a form on $2l+1$ variables, but not into a form on $2l$ variables, are that every pfaffian of order $>2l$ vanishes and that not every principal semi-minor $\{i_1 i_2 \cdots i_{2l+1}\}$ of order $2l+1$ vanishes.

Next, let every $\{1 \cdots 2l\, i\} = 0$. Then by a linear transformation of determinant unity, (37) can be transformed* into

$$\sum_{s=1}^{l} x_{2s-1}x_{2s} + x_1^2 + \delta x_2^2, \qquad \delta \equiv \sum_{s=1}^{l} \delta_{2s-1}\delta_{2s}. \tag{41}$$

Now $x_1x_2 + x_1^2 + \delta x_2^2$ is reducible or irreducible in the $GF[2^n]$ according as $\chi(\delta) = 0$ or 1, where

$$\chi(\delta) = \sum_{i=0}^{n-1} \delta^{2^i} \qquad \chi^2 \equiv \chi. \tag{42}$$

The forms (41) with $\chi(\delta) = 0$ are all equivalent to

$$\sum_{s=1}^{l} x_{2s-1}x_{2s} \tag{43}$$

under the group G_1 of linear transformations of determinant unity. The forms (41) with $\chi(\delta) = 1$ are equivalent under G_1, but no one of them can be transformed into (43). The reduction of Q_m to (37) was effected by a transformation of determinant (39). If $m > 2l$, Q_m can be transformed into (41) within G_1; but if $m = 2l$, the canonical form within G_1 may be taken to be (41) with one of the variables multiplied by $[1 \cdots 2l]$. Among the results established, we mention the following:

Necessary and sufficient conditions that Q_m shall be transformable into a form on $2l$ variables, but not into a form on $2l-1$ variables, are that every pfaffian of order $>2l$ and every $\{i_1 \cdots i_{2l+1}\}$ shall vanish, but not every pfaffian of order $2l$.

13. A quadratic form in the $GF[2^n]$ may be said to be of *rank* r if, of the principal minors of even order and the algebraic halves of the principal minors of odd order, those of order $>r$ vanish, but not all of order r vanish. As an equivalent definition we may say that Q_m is of odd rank $2l+1$ if every† $[i_1 \cdots i_{2l+2}]$ but not every $\{i_1 \cdots i_{2l+1}\}$ vanishes; that Q_m is of even rank $2l$ if every $[i_1 \cdots i_{2l+2}]$ and every‡ $\{i_1 \cdots i_{2l+1}\}$ but not every $[i_1 \cdots i_{2l}]$ vanishes.

In comparing these definitions with the algebraic definition of the rank of a quadratic form or symmetrical matrix, we note that a principal minor of odd

*American Journal of Mathematics, vol. 30 (1908), p. 266, § 6.

† Then every pfaffian of order $>2l$ vanishes, also every $\{i_1 \cdots i_t\}$, $t > 2l+2$, by (33), (34).

‡ The vanishing of these does not imply that of the pfaffians of order $\geqq 2l+2$, as shown by the example $x_1x_2 + x_3x_4 + \cdots + x_{2l+1}x_{2l+2}$.

order vanishes identically modulo 2, so that we have introduced their algebraic halves; furthermore, when a principal minor M of even order vanishes in the $GF[2^n]$, all the first minors * M_{ij} of M also vanish, since $M_{ii}M_{jj} - M_{ij}^2$ is a multiple of M and since M_{ii} is multiple of 2.

From the results stated in italics in § 12 we have the

THEOREM. *A quadratic form in the $GF[2^n]$ can be transformed into a form on r variables, but not into one on $r-1$ variables, if and only if its rank is r.*

Second method of reduction of a quadratic form.

14. We consider a second method † of normalizing a quadratic form (24) in the $GF[2^n]$. If $[1\cdots m] \neq 0$, so that m is an even number $2l$, Q_m is already of the form (50). In the contrary case there is some integer $l < m/2$ such that every pfaffian of order $> 2l$, but not every pfaffian of order $2l$, vanishes. We assume that $l > 0$, thus excluding the rather trivial case in which every $\beta_{ij} = 0$. After rearranging the variables, we may set $[1\cdots 2l] \neq 0$. To Q_m we apply the transformation

$$x_i = x_i' + c_i x_m' \ (i = 1, \cdots, 2l), \qquad x_i = x_i' \ (i = 2l+1, \cdots, m).$$

Let $\beta_{ji} = \beta_{ij}$, $\beta_{ii} = 0$. Then the resulting form is

$$(44) \qquad \sum_{i<j}^{1,\cdots,m-1} \beta_{ij} x_i' x_j' + \sum_{j=1}^{m-1} B_{jm} x_j' x_m' + \sum_{i=1}^{m-1} b_i x_i'^2 + E_m x_m'^2,$$

$$(45) \quad B_{jm} = \sum_{i=1}^{2l} \beta_{ij} c_i + \beta_{jm}, \qquad E_m = \sum_{j=1}^{2l} \beta_{jm} c_j + \sum_{i<j}^{1,\cdots,2l} \beta_{ij} c_i c_j + \sum_{i=1}^{2l} b_i c_i^2 + b_m.$$

By choice of the c_i, we may make $B_{jm} = 0$ $(j = 1, \cdots, 2l)$. In fact, $[1\cdots 2l]^2 c_i$ equals the minor of β_{im} in the skew-symmetric determinant giving the square of $[1\cdots 2l\,m]$. Apart from sign, this minor equals ‡

$$[1\cdots i-1\ i+1\cdots 2l\,m][1\cdots 2l].$$

Hence

$$(46) \qquad c_i = [1\cdots 2l]^{-1}[1\cdots i-1\ i+1\cdots 2l\,m] \qquad (i = 1, \cdots, 2l).$$

For these values of the c_i, we have $B_{jm} = 0$ for any j, as may be shown from (46) or without computation as follows. The determinant of the coefficients of

* As shown earlier, the introduction of the $\frac{1}{2}M_{ii}$ serves to define the rank. Contrary to the suggestions in my earlier papers I now prefer to avoid the use of semi-minors other than principal, since $\frac{1}{2}M_{ij}$ $(i \neq j)$ is defined in the field only when $M = 0$, and even then in a very artificial manner.

† Having points of resemblance and points of contrast with KRONECKER's algebraic method (§ 7).

‡ SCOTT, *Theory of Determinants*, 1880, p. 75, § 15.

$c_1, \cdots, c_{2l}, 1$ in B_{jm} $(j = 1, \cdots, 2l, k)$ is the minor M_{km} of the element β_{km} in the last row of

$$D_{2l+2} = |\beta_{st}| \qquad (s, t = 1, \cdots, 2l, k, m).$$

The latter equals $[1 \cdots 2l\, k\, m]^2$ and is zero by hypothesis. Let M_{ii} denote the minor of β_{ii}, so that $M_{ii} = 0$. But $M_{kk}M_{mm} - M_{km}^2$ equals the product of D_{2l+2} by the minor $|\beta_{st}|$, $s, t = 1, \cdots, 2l$. Hence $M_{km} = 0$.

To evaluate E_m, we note that the preceding discussion is valid for any m. For $m = 2l + 1$, the semi-discriminant of (44), with each $B_{jm} = 0$, equals $[1 \cdots 2l]^2 E_{2l+1}$, while that of Q_{2l+1} is $\{1 \cdots 2l + 1\}$. Now the first sum in E_m, given by (45), becomes a multiple of 2 when β_{jm} is eliminated by means of $B_{jm} = 0$. Hence

$$(47) \qquad \{1 \cdots 2l + 1\} = [1 \cdots 2l]^2 \left\{ \sum_{i<j}^{1, \cdots, 2l} \beta_{ij} c_i c_j + \sum_{i=1}^{2l} b_i c_i^2 + b_{2l+1} \right\}$$

becomes an identity when the values (46), with $m = 2l + 1$, are inserted:

$$(48) \qquad \begin{aligned} \{1 \cdots 2l + 1\} \equiv & \sum_{i<j}^{1, \cdots, 2l} \beta_{ij} [1 \cdots i-1\, i+1 \cdots 2l+1] \\ & \times [1 \cdots j-1\, j+1 \cdots 2l+1] + \sum_{i=1}^{2l+1} b_i [1 \cdots i-1\, i+1 \cdots 2l+1]^2. \end{aligned}$$

Replacing the subscript $2l + 1$ by m, we derive the identity

$$(49) \qquad \{1 \cdots 2l\, m\} \equiv [1 \cdots 2l]^2 E_m.$$

For the next step, we add suitable multiples of x_{m-1} to $x_1, \cdots, x_{2l}$. After $m - 2l$ such steps, we find that Q_m has been reduced, by a transformation of determinant unity, to

$$(50) \qquad \sum_{i<j}^{1, \cdots, 2l} \beta_{ij} x_i x_j + \sum_{i=1}^{2l} b_i x_i^2 + [1 \cdots 2l]^{-2} \sum_{i=2l+1}^{m} \{1 \cdots 2l\, i\} x_i^2.$$

The first two sums define Q_{2l}, viz., (24) for $m = 2l$.

15. For the further normalization of (50) we shall apply transformations involving only the variables $x_1, \cdots, x_{2l}$. In particular, the final sum in (50) will not be altered. The present problem is therefore the normalization of Q_{2l} of discriminant $[1 \cdots 2l]^2 \neq 0$. Thus not every pfaffian of order $2l - 2$ vanishes; let $[1 \cdots 2l - 2] \neq 0$. We may proceed as in § 14 (with m replaced by $2l$, and l by $l - 1$) with a certain essential modification.* The transformation

$$x_i = x_i' + [1 \cdots 2l-2]^{-1} [1 \cdots i-1\, i+1 \cdots 2l-2\, 2l]\, x_{2l}' \quad (i = 1, \cdots, 2l-2),$$

$$x_{2l-1} = x_{2l-1}', \qquad x_{2l} = x_{2l}',$$

* In the proof that $B_{jm} = 0\,(j > 2l)$ by means of the vanishing of the pfaffians of order $2l + 2$. The latter correspond to pfaffians of order $2l$ in the present case, and these are not all zero. Instead of the B_{jm} we now have B_i, which does not vanish.

of determinant unity, replaces Q_{2l} by

$$(51)\qquad \sum_{i<j}^{1,\cdots,2l-1}\beta_{ij}x_i'x_j'+Bx_{2l-1}'x_{2l}'+\sum_{i=1}^{2l-1}b_ix_i'^2+[1\cdots 2l-2]^{-2}\{1\cdots 2l-2\;2l\}x_{2l}'^2,$$

where

$$B_l\equiv\sum_{i=1}^{2l-2}\beta_{i\,2l-1}[1\cdots 2l-2]^{-1}[1\cdots i-1\;i+1\cdots 2l-2\;2l]-\beta_{2l-1\,2l}$$
$$=[1\cdots 2l-2]^{-1}[1\cdots 2l].$$

The last equality follows from the expansion of $[1\cdots 2l]$ with respect to the elements $\beta_{i\,2l-1}$, or by the identity of the discriminants of Q_{2l} and (51).

Similarly, to (51) we apply the transformation

$$x_i'=x_i''+[1\cdots 2l-2]^{-1}[1\cdots i-1\;i+1\cdots 2l-2\;2l-1]x_{2l-1}''\quad(i=1,\cdots,2l-2),$$
$$x_{2l-1}'=x_{2l-1}'',\qquad x_{2l}'=x_{2l}'',$$

suppress the accents on x'', and obtain the form

$$(52)\qquad \sum_{i<j}^{1,\cdots,2l-2}\beta_{ij}x_ix_j+[1\cdots 2l-2]^{-1}[1\cdots 2l]x_{2l-1}x_{2l}+\sum_{i=1}^{2l-2}b_ix_i^2$$
$$+[1\cdots 2l-2]^{-2}\{1\cdots 2l-2\;2l-1\}x_{2l-1}^2+[1\cdots 2l-2]^{-2}\{1\cdots 2l-2\;2l\}x_{2l}^2.$$

The first and third sums define Q_{2l-2}. The double step by which Q_{2l} has been reduced to (52) by a transformation of determinant unity may now be repeated. By rearranging the variables $x_1,\cdots,x_{2l-2}$, we may assume that (28) holds. We arrive ultimately at the following conclusion: If every pfaffian of order $>2l$, but not every pfaffian of order $2l$, vanishes, the pfaffians $[12]$, $[1234],\cdots,[1\cdots 2l]$ may be assumed not to vanish; then Q_m may be reduced by a linear transformation of determinant unity to

$$(53)\quad \sum_{s=1}^{l}[1\cdots 2s-2]^{-1}[1\cdots 2s]x_{2s-1}x_{2s}+\sum_{s=1}^{l}[1\cdots 2s-2]^{-2}(\{1\cdots 2s-2\;2s-1\}x_{2s-1}^2$$
$$+\{1\cdots 2s-2\;2s\}x_{2s}^2)+[1\cdots 2l]^{-2}\sum_{i=2l+1}^{m}\{1\cdots 2l\,i\}x_i^2.$$

If we multiply x_{2s-1} and x_{2s} by $[1\cdots 2s-2]$, for $s=1,\cdots,l$; and x_i by $[1\cdots 2l]$, for $i=2l+1,\cdots,m$, we see that Q_m can be reduced by a linear transformation of determinant (35) to the form (36).

Hence the present method of reduction has led us to the same normal form as that obtained by the former method.

Definition and construction of the invariant χ_l.

16. It remains to investigate a problem of decided importance both for the general theory of the reduction of quadratic forms in the $GF[2^n]$ and for the

subsequent determination of the invariants. Under the assumption (28) that no one of the pfaffians $[12]$, $[1234]$, $\cdots$, $[1\cdots 2l]$ vanishes, the form Q_{2l} was transformed into the normal form (41), in which

$$(54)\quad \delta=\sum_{s=1}^{l}[1\cdots 2s-2]^{-2}[1\cdots 2s]^{-2}\{1\cdots 2s-2\,2s-1\}\{1\cdots 2s-2\,2s\}.$$

Of the forms (41), there are two non-equivalent canonical forms distinguished by the value 0 or 1 of $\chi(\delta)$, where χ is the function (42). Given a form Q_{2l} having $[1\cdots 2l]\neq 0$, we readily obtain by a suitable rearrangement of the variables a form Q'_{2l} for which the assumption (28) holds, so that the criterion $\chi(\delta)=0$ or 1 is applicable. What we desire, however, is a criterion which will apply directly to Q_{2l} itself. Moreover, we prefer to accomplish this result by means of an absolute invariant* of Q_{2l}, which must therefore be defined for all values of the coefficients including those making $[1\cdots 2l]=0$. Hence we seek a polynominal χ_l in the coefficients of Q_{2l} such that $\chi_l=1$ when Q_{2l} is transformable into (41) with $\chi(\delta)=1$, while $\chi_l=0$ for all remaining forms Q_{2l}. Such a polynominal χ_l is clearly an absolute invariant of Q_{2l}.

For $l=1$, $\beta_{12}\neq 0$, (54) gives $\delta=b_1b_2\beta_{12}^{-2}$. Hence we have

$$(55)\qquad \chi_1=\chi(b_1b_2\beta_{12}^{2^n-3}),$$

where (and below) an exponent 2^n-3 is to be replaced by unity when $n=1$.

Next, let $l=2$. For $\beta_{12}\neq 0$, $P\equiv[1234]\neq 0$, (54) gives

$$\delta=\beta_{12}^{-2}b_1b_2+\beta_{12}^{-2}P^{-2}\{123\}\{124\}.$$

Thus $\delta=\delta_{12}$, where

$$(56)\qquad \delta_{ij}=(b_ib_jP^2+\{ijk\}\{ijt\})(\beta_{ij}P)^{2^n-3},$$

i,j,k,t being a permutation of 1, 2, 3, 4. Hence

$$(57)\qquad \chi_2=\chi(\delta_{ij})$$

for every set of values of the 10 coefficients of Q_4 for which $\beta_{ij}\neq 0$ (whether or not $P\neq 0$). In particular, we deduce the identity

$$(58)\qquad \chi_2\equiv\chi(\delta_{12})+M(\beta_{12}^{\nu}-1)\qquad(\nu=2^n-1).$$

Consider any set with $\beta_{12}=0$, $\beta_{13}\neq 0$. By (58), and (57) for $i,j=1,3$,

$$[\chi(\delta_{13})]_{\beta_{12}=0}=M\qquad(\text{for }\beta_{13}\neq 0)$$
$$\equiv M+M_1(\beta_{13}^{\nu}-1),$$

where M_1 is free of β_{12}, β_{13}. Then (58) becomes

$$(59)\qquad \chi_2\equiv\chi(\delta_{12})+(\beta_{12}^{\nu}-1)\chi(\delta_{13})+M_1(\beta_{12}^{\nu}-1)(\beta_{13}^{\nu}-1).$$

* For the $GF[p^n]$, $p>2$, we used the power $\frac{1}{2}(p^n-1)$ of the discriminant.

Consider any set with $\beta_{12}=\beta_{13}=0$, $\beta_{14}\neq 0$. By (59) and (57) for $i,j=1,4$,

$$[\chi(\delta_{14})]_{\beta_{12}=\beta_{13}=0}\equiv M_1+M_2(\beta_{14}^{r}-1),$$

$$\chi_2\equiv\chi(\delta_{12})+(\beta_{12}^{r}-1)\chi(\delta_{13})+(\beta_{12}^{r}-1)(\beta_{13}^{r}-1)\chi(\delta_{14}) \tag{60}$$
$$+M_2(\beta_{12}^{r}-1)(\beta_{13}^{r}-1)(\beta_{14}^{r}-1),$$

where M_2 is free of $\beta_{12}, \beta_{13}, \beta_{14}$. If these β's vanish, $P=0$, so that $\chi_2=0$ by definition. Hence (60) gives $M_2=0$. Thus

$$\chi_2=\chi(e), \qquad e=\delta_{12}+(\beta_{12}^{r}-1)\delta_{13}+(\beta_{12}^{r}-1)(\beta_{13}^{r}-1)\delta_{14}. \tag{61}$$

The terms independent of the b's in χ_2 are

$$\chi(\beta_{12}^{r}\beta_{13}\beta_{14}\beta_{23}\beta_{24}P^{2^n-3})\equiv\chi\{(\beta_{13}\beta_{14}\beta_{23}\beta_{24}+\beta_{12}\beta_{13}\beta_{24}\beta_{34}+\beta_{12}\beta_{14}\beta_{23}\beta_{34})P^{2^n-3}\}.$$

For $\beta_{12}\neq 0$, this identity may be written

$$\chi\{t(P-t)P^{2^n-3}\}=0, \qquad t\equiv\beta_{12}\beta_{34},$$

and is true since the square of tP^{2^n-2} equals $t^2P^{2^n-3}$. For $\beta_{12}=0$, the identity reduces to

$$\chi\{\alpha\beta(\alpha+\beta)^{2^n-3}\}=0, \qquad \alpha\equiv\beta_{13}\beta_{24}, \qquad \beta\equiv\beta_{14}\beta_{23}.$$

In view of (55), it expresses the condition that the quadratic form

$$x^2+(\alpha+\beta)xy+\alpha\beta y^2\equiv(x+\alpha y)(x+\beta y)$$

shall be reducible in the field.

In e the coefficients of b_3b_4, b_4, b_4^2 are the products of P^{2^n-3} by β_{12}^2, $\beta_{12}\beta_{13}\beta_{23}$, 0, respectively. Hence, by the symmetry of an invariant, we have*

$$\chi_2=\chi\{(\sum_{(3)}\beta_{13}\beta_{14}\beta_{23}\beta_{24}+\sum_{(4)}\beta_{12}\beta_{13}\beta_{23}b_4+\sum_{(6)}\beta_{12}^2b_3b_4)P^{2^n-3}\}. \tag{62}$$

For a general value of l, we shall express χ_l in terms of χ_{l-1}. We develop an auxiliary formula for the case $[1\cdots 2l-2]\neq 0$, $P\equiv[1\cdots 2l]\neq 0$. Then, by §15, Q_{2l} can be transformed into (52). Multiplying x_{2l-1} by $[1\cdots 2l-2]$ and x_{2l} by P^{-1}, we get

$$Q_{2l-2}+x_{2l-1}x_{2l}+\{1\cdots 2l-2\,2l-1\}x_{2l-1}^2+[1\cdots 2l-2]^{-2}P^{-2}\{1\cdots 2l-2\,2l\}x_{2l}^2.$$

First, let $\chi_{l-1}=0$. Then, by the proof leading to (43), Q_{2l-2} can be transformed into $\sum_{s=1}^{s=l-1}x_{2s-1}x_{2s}$. Hence $\chi_l=\chi(\delta)$, where δ equals

$$\delta_{1,\ldots,2l-2}=\{1\cdots 2l-2\,2l-1\}\{1\cdots 2l-2\,2l\}[1\cdots 2l-2]^{2^n-3}P^{2^n-3}. \tag{63'}$$

Next, let $\chi_{l-1}=1$. Then Q_{2l-2} can be transformed into

$$\sum_{s=1}^{l-1}x_{2s-1}x_{2s}+x_1^2+dx_2^2, \qquad \chi(d)=1.$$

Hence $\chi_l=\chi(d+\delta)=1+\chi(\delta)$, where δ is defined by (63').

*American Journal of Mathematics, loc. cit., p. 267, §7, for the special case $P=1$.

Thus, in either case, $\chi_l = \chi_{l-1} + \chi(\delta)$. Hence for any P,

$$\chi_l = \chi_{l-1} + \chi(\delta) + \mu(P^v - 1) \qquad \text{when} \qquad [1 \cdots 2l-2] \neq 0.$$

Let $P = 0$. Then $\chi_l = 0$ by definition. Thus $\mu = \chi_{l-1}$, and

$$(64') \qquad \chi_l = P^v \chi_{l-1} + \chi(\delta_{1,\ldots,2l-2}) \qquad \text{when} \qquad [1 \cdots 2l-2] \neq 0.$$

Let $i_1, \cdots, i_{2l}$ give any permutation of $1, \cdots, 2l$. Then

$$(64) \qquad \chi_l = P^v \chi_{l-1} + \chi(\delta_{i_1,\ldots,i_{2l-2}}) \qquad \text{when} \qquad [i_1 \cdots i_{2l-2}] \neq 0,$$

$$(63) \qquad \delta_{i_1,\ldots,i_{2l-2}} = \{i_1 \cdots i_{2l-2} i_{2l-1}\}\{i_1 \cdots i_{2l-2} i_{2l}\}[i_1 \cdots i_{2l-2}]^{2^r-3} P^{2^r-3}.$$

By our usual synthesis, we obtain *

$$(65) \qquad \begin{aligned} \chi_l = {} & P^v \chi_{l-1} + \chi(d_1) + (P_1^v - 1)\chi(d_2) \\ & + (P_1^v - 1)(P_2^v - 1)\chi(d_3) + \cdots + (P_1^v - 1)\cdots(P_{\lambda-1}^v - 1)\chi(d_\lambda), \end{aligned}$$

where P_j denotes the pfaffian $[i_1 \cdots i_{2l-2}]$, d_j the corresponding function (63), λ the number $l(2l-1)$ of such pfaffians, and $P_1, \cdots, P_\lambda$ these pfaffians in any sequence.

For $l = 3$, set $P_1 = [1234]$, $P_2 = [1235]$, $P_3 = [1236]$. Now

$$\pi = (P_1^v - 1)(P_2^v - 1)(P_3^v - 1)P$$

is a factor of the terms of (65) after the fourth. But

$$(66) \qquad P_1[1256] - P_2[1246] + P_3[1245] \equiv \beta_{12} P,$$

algebraically. Hence $\pi = 0$ if $\beta_{12} \neq 0$. Interchanging 2 and 3, or 1 and 3 in (66), we find that $\pi = 0$ unless $\beta_{12} = \beta_{13} = \beta_{23} = 0$. In the latter case, $P_1 = P_2 = P_3 = 0$. Then let $P_4, \cdots, P_{12}$ denote the pfaffians $[ijkt]$ in which i, j are chosen from 1, 2, 3, and k, t from 4, 5, 6. Then each d_j $(j = 4, \cdots, 12)$ has the factor $\{123kt\}$, which is a linear homogeneous function of b_1, b_2, b_3. Among the P_j $(j \leqq 12)$ occurs every $[1r3s]$. Interchanging 2 with r in (66) we conclude, as above, that the terms of (65) with the factor $P\Pi(P_j^v - 1)$, $j = 1, \cdots, 12$, are zero; this is evident if each $\beta_{1r} = 0$, whence $P = 0$. Hence $\chi_3 = P^v \chi_2 + \chi(E)$, where

$$E = \delta_{1234} + (P_1^v - 1)\delta_{1235} + (P_1^v - 1)(P_2^v - 1)\delta_{1236} + L,$$

every term of L containing b_1, b_2, or b_3. In E the coefficients of $b_5 b_6$, b_6^2, b_6 are the products of P^{2^n-3} by

$$[1234]^2, \qquad 0, \qquad V \equiv P_1^v \psi + (P_1^v - 1)P_2^v \psi,$$

* The argument initially gives an additional term πM, where π is the product of the $P_j^v - 1$ for $j = 1, \cdots, \lambda$. But if we set every $P_j = 0$, we have $P = 0$, so that $\chi_l = 0$, by definition. Thus $0 = M$.

respectively, where ψ denotes the aggregate ψ_{12345} of the terms free of the b's in $\{12345\}$. Evidently $V=\psi$ if either $P_1 \neq 0$ or $P_2 \neq 0$. Since χ_3 is an absolute invariant, V is symmetrical in $1, \cdots, 5$. Hence $V=\psi$ unless every $[i_1 i_2 i_3 i_4]=0$, where $i, \cdots, i_4$ are chosen from $1, \cdots, 5$. But in the latter case, $\psi=0$, by (48), while V obviously vanishes. Hence

$$\chi_3 = \chi\{(F_3 + \Sigma \psi_{12345} b_6 + \Sigma [1234]^2 b_5 b_6) P^{2^n-3}\}, \tag{67}$$

where F_3 may be taken to be

$$F_3 = \sum_{(45)} \beta_{13}\beta_{14}\beta_{23}\beta_{24}\beta_{56}^2 + \sum_{(10)} \beta_{12}\beta_{13}\beta_{23}\beta_{45}\beta_{46}\beta_{56}.$$

An inspection of the expressions (55), (62), (67) for χ_l, for $l=1, 2, 3$, indicates that, in the formula, given by (65) and (63),

$$\chi_l = \chi(CP^{2^n-3}), \tag{68}$$

C has the following simple relation to the algebraic discriminant D of the quadratic form Q_{2l} (§ 10). The coefficients of $4b_{2l-1}b_{2l}$ in $4C$ and D are congruent modulo 2, likewise the coefficients* of $4b_{2l}$; while the terms independent of the b's in $D-(-1)^l P^2$ and $4C$ are multiples of 4, congruent modulo 8. In the last statement and in the proofs below, P is the expression obtained from the algebraic pfaffian $[1 \cdots 2l]$ by giving the various terms any desired signs. For such a function P, the congruence

$$D-(-1)^l P^2 \equiv 4C \pmod 8 \tag{69}$$

uniquely determines $C \pmod 2$. The resulting function (68) will be shown to be an absolute invariant of Q_{2l} in the $GF[2^n]$. Under any linear transformation of determinant Δ, D becomes $\Delta^2 D$ and P becomes $\Delta P + 2\rho$ (§ 10). Hence $d=D-(-1)^l P^2$ becomes $\Delta^2 d-(-1)^l(4\Delta P\rho + 4\rho^2)$. But, by (69), d is a multiple of 4. Hence dP^{2^n-3} takes the increment

$$(\Delta P)^{2^n-3}(4\Delta P\rho + 4\rho^2) + 8S.$$

Hence in the $GF[2^n]$, the function (68) takes the increment

$$\chi[(\Delta P)^{2^n-2}\rho] + \chi[(\Delta P)^{2^n-3}\rho^2] \equiv 0,$$

since the quantity in the second brackets is the square of that in the first.

For the canonical form (41), $D=(-1)^l(1-4\delta)$, $P^2=1$. Hence

$$C \equiv \delta, \qquad \chi_l \equiv \chi(\delta) \pmod 2,$$

so that (68) is the desired absolute invariant of Q_{2l}.

*Such terms, containing a single b, occur when $l>1$.

The invariants of a quadratic form Q_m in the $GF[2^n]$, §§ 17–20.

17. Under the group G of all m-ary linear homogeneous transformations in the $GF[2^n]$, we define C_0 to be the class to which the identically vanishing form Q_m belongs, C_{2l+1} the class for (40) with $c=1$, $C_{2l,1}$ the class for (41) with $\chi(\delta)=1$, $C_{2l,0}$ the class for (41) with $\chi(\delta)=0$.

We seek the characteristic absolute invariants I_0, I_{2l+1}, $I_{2l,1}$, $I_{2l,0}$ of the general Q_m for the respective classes. By (1),

$$I_0=\prod_{i<j}(\beta_{ij}^{\nu}-1)\cdot\prod(b_i^{\nu}-1) \qquad (\nu=2^n-1). \tag{70}$$

We shall employ the abbreviations

$$\pi_{2l}=\Pi(P^{\nu}-1), \qquad \sigma_{2l+1}=\Pi(S^{\nu}-1), \tag{71}$$

where P ranges over the pfaffians $[i_1\cdots i_{2l}]$ of order $2l$, while S ranges over the principal semi-minors $\{i_1\cdots i_{2l+1}\}$ of order $2l+1$. In particular,

$$\pi_2=\prod_{i<j}(\beta_{ij}^{\nu}-1), \qquad \sigma_1=\prod(b_i^{\nu}-1), \tag{71'}$$

$$I_0=\pi_2\sigma_1. \tag{70'}$$

Since $I_{2l+1}=1$ if every pfaffian of order $2l+2$ vanishes, but not every principal semi-minor $\{i_1\cdots i_{2l+1}\}$, while $I=0$ in the remaining cases, we have

$$I_{2l+1}=\pi_{2l+2}(1+\sigma_{2l+1}), \tag{72}$$

$$I_1=\pi_2+I_0. \tag{72'}$$

We readily determine the absolute invariant

$$R_{2l}=I_{2l,1}+I_{2l,0}, \tag{73}$$

which has the value 1 or 0 according as Q_m is or is not of rank $2l$. By the result at the end of § 12, we have

$$R_{2l}=\pi_{2l+2}\sigma_{2l+1}(1+\pi_{2l}), \tag{74}$$

the first two factors being absent if $2l=m$, so that

$$R_m=[1\cdots m]^{\nu} \qquad (m \text{ even}). \tag{74'}$$

By the same reference, we have

$$I_{2l,1}=\pi_{2l+2}\sigma_{2l+1}L, \tag{75}$$

where L is to be determined. Henceforth we consider only sets of coefficients for which every pfaffian of order $2l+2$ and every $\{i\cdots i_{2l+1}\}$ vanishes; then $I_{2l,1}=L$. Consider such a set with $[1\cdots 2l]\neq 0$; then Q_m can be transformed into the corresponding form Q_{2l} on $x_1,\cdots,x_{2l}$ (§ 14), and L has the

value χ_l defined by (68). Let the various pfaffians of order $2l$ be designated by $P_1, \cdots, P_t$. When $1, \cdots, 2l$ are replaced by $i_1, \cdots, i_{2l}$, $P = [1 \cdots 2l]$ becomes $P_i = [i_1 \cdots i_{2l}]$; let the function C become C_i. Then for any one of the above sets for which $P_i \neq 0$, L has the value

$$L_i = \chi(C_i P_i^{2^n-3}). \tag{76}$$

By the usual synthesis of these relations,

$$L = L_1 + (P_1^\nu - 1)L_2 + (P_1^\nu - 1)(P_2^\nu - 1)L_3 + \cdots + (P_1^\nu - 1)\cdots(P_{t-1}^\nu - 1)L_t, \tag{77}$$

with initially an addition term $M\Pi(P_i^\nu - 1)$, $i = 1, \cdots, t$. But for every $P_i = 0$, $I_{2l,1}$ is zero, by definition, so that $M = 0$. Similarly,

$$I_{2l,0} = \pi_{2l+2}\sigma_{2l+1}K.$$

For the sets with every $[i_1 \cdots i_{2l+2}] = \{i_1 \cdots i_{2l+1}\} = 0$, we have $I_{2l,0} = K$. If also $P_i \neq 0$, then $K = 1 + L_i$. A synthesis of the latter relations gives

$$K = 1 + L + \mu\prod_{i=1}^{t}(P_i^\nu - 1).$$

If every $P_i = 0$, then $I_{2l,0} = 0$, by definition. Hence $\mu = 1$. Thus

$$I_{2l,0} = \pi_{2l+2}\sigma_{2l+1}(L + 1 + \pi_{2l}). \tag{78}$$

As a check, we note that (74) follows at once from (75) and (78); also that for $m = 3$, $I_{2,1}$ is the invariant F given for $n \leqq 4$ in the American Journal of Mathematics, l. c.

18. The preceding determination of the invariant $I_{2l,1}$ was based upon the second method (§ 14) of reducing quadratic forms in the $GF[2^n]$. For $l > 1$, a determination based upon the first method (§ 12) appears to be more complicated. We shall treat briefly the case $l = 1$, $m = 4$, from the latter standpoint; we are thereby led naturally to a noteworthy modification of the earlier formula for $I_{2,1}$. If $\beta_{12} \neq 0$, Q_4 is transformable into an irreducible binary form $B = x_1x_2 + x_1^2 + cx_2^2$ if and only if $P \equiv [1234] = 0$, $\{123\} = 0$, $\{124\} = 0$, and (55) is unity. In general, if $\beta_{ij} \neq 0$, $I_{2,1} = 1$ if and only if $P = 0$, $\{ijk\} = 0$, $\{ijt\} = 0$, $L_{ij} = 1$, where i, j, k, t form a permutation of 1, 2, 3, 4, and

$$L_{ij} = \chi(b_i b_j \beta_{ij}^{2^n-3}), \qquad G_{ij} = (\{ijk\}^\nu - 1)(\{ijt\}^\nu - 1)L_{ij}. \tag{79}$$

Hence, if $\beta_{ij} \neq 0$, $I_{2,1} = (P^\nu - 1)G_{ij}$. By the usual synthesis,

$$\begin{aligned} I_{2,1} = (P^\nu - 1)\{G_{12} + (\beta_{12}^\nu - 1)G_{13} + (\beta_{12}^\nu - 1)(\beta_{13}^\nu - 1)G_{14} \\ + \cdots + (\beta_{12}^\nu - 1)\cdots(\beta_{24}^\nu - 1)G_{34}\}. \end{aligned} \tag{80}$$

A comparison of (80) with (75) and (77) indicates that

$$(P^\nu - 1)G_{ij} = (P^\nu - 1)\sigma_3 L_{ij}, \qquad \sigma_3 \equiv \prod_{(4)}(\{ijk\}^\nu - 1).$$

This equality is obvious except in the case $P=0$, $\{ijk\}=0$, $\{ijt\}=0$, $L_{ij}=1$. But then Q_4 is transformable into B, so that $\{ikt\}=0$, $\{jkt\}=0$.

19. To determine the relative invariants of Q_m in the $GF[2^n]$, we note that (end of § 12) the only classes under the total group G which separate into sub-classes under the group G_1 of determinant unity are $C_{m,1}$ and $C_{m,0}$ for m even, while the only such class is C_m for m odd. In the latter case, the sub-classes may be designated by $K_{m,c}$, a representative form being (40) for $m = 2l+1$. Here c is the non-vanishing semi-discriminant of Q_m. As in § 9, there exists a single invariant of a given weight w, $0 < w < \nu$, since it must vanish for all the classes other than the $K_{m,c}$, and since its value for $K_{m,c}$ must be $c^{w/2}$ times its value $\neq 0$ for $K_{m,1}$. But such an invariant is $S^{w/2}$ for w even and $S^{(w+\nu)/2}$ for w odd, where S is the semi-discriminant of the general form Q_m. *Hence for m odd, the only relative invariants are powers of the semi-discriminant.*

Next let m be even. The sub-classes mentioned above may be designated $K_{m,\chi,P}$, where $\chi = 1$ or 0, and $P = [1\cdots m] \neq 0$, representative forms being

$$(41') \qquad \sum_{s=2}^{\frac{1}{2}m} x_{2s-1}x_{2s} + Px_1x_2 + P^2x_1^2 + \delta x_2^2 \qquad [\chi(\delta)=\chi].$$

To obtain an invariant ϕ of weight w, $0 < w < \nu$, we must assign to ϕ the value 0 for all the classes other than the $K_{m,\chi,P}$, and for the latter P^w times the value of ϕ for $K_{m,\chi,1}$. Hence the two values for the last two classes ($\chi = 1$ or 0) alone are arbitrary, so that there are just two linearly independent invariants of the given weight w. But P^w and $P^w\chi_{m/2}$ have these properties, $\chi_{m/2} = I_{m,1}$ being the absolute invariant determined in § 16. *For m even, the only relative invariants are*

$$[1\cdots m]^w(c_1 + c_2\chi_{m/2}) \qquad (0 < w < 2^n - 1,\ c_1 \text{ and } c_2 \text{ constants}).$$

20. A simple method of obtaining a smaller number of independent invariants of Q_m in the $GF[2^n]$, in terms of which all the above invariants can be expressed rationally, will be illustrated by the case $m = 4$. Let

$$(81) \qquad F_4 = I_{4,1} + I_3, \qquad J_4 = I_{2,1} + I_0, \qquad P = [1234].$$

By (72) for $l = 1$, and (75) for $l = 2$,

$$(82) \quad I_3 = (P^\nu - 1)(1 + \sigma_3), \qquad I_{4,1} = L \equiv \chi_2, \qquad \sigma_3 \equiv \prod_{(4)}(S^\nu - 1).$$

Since χ_2, given by (62), is a multiple of P, $(P^\nu - 1) I_{4,1} = 0$. Hence *

(83) $\chi_2 = I_{4,1} = P^\nu F_4, \qquad I_3 = (P^\nu - 1) F_4, \qquad I_{4,0} = P^\nu (F_4 + 1),$

the last following from $I_{4,1} + I_{4,0} = P^\nu$, given by (73) and (74′) for $l = 2$. By (73) and (74) for $l = 1$,

$$I_{2,1} + I_{2,0} = (P^\nu - 1)\sigma_3(1 + \pi_2) = (P^\nu - 1)\sigma_3 + \pi_2,$$

since each $\pi_2 S = 0$, $\pi_2 P = 0$, in view of the form $(71_1')$ of π_2. But by (82_1) and (83_2),

$$(P^\nu - 1)\sigma_3 = I_3 + P^\nu - 1 = (P^\nu - 1)(F_4 + 1).$$

Hence we have†

(84) $$I_{2,1} + I_{2,0} = (P^\nu - 1)(F_4 + 1) + \pi_2.$$

Now each term of (80) has a factor β_{ij}, in view of the L_{ij}. Thus

$$\pi_2 I_{2,1} = 0, \qquad \pi_2 I_0 = I_0,$$

the latter following from (70′). Hence by (81_2)

(85) $$I_0 = \pi_2 J_4, \qquad I_{2,1} = J_4(\pi_2 + 1).$$

Then by (72′) and (84),

(86) $I_1 = \pi_2(J_4 + 1), \qquad I_{2,0} = (P^\nu - 1)(F_4 + 1) + J_4(\pi_2 + 1) + \pi_2.$

Hence formulæ (83), (85), (86) express all the absolute invariants of Q_4 in terms of F_4, J_4, P, π_2. Incorporating the result at the end of § 19, we have the

THEOREM.‡ *Every invariant of Q_4 in the $GF[2^n]$ is an integral function of the invariants F_4, J_4, P, π_2; in fact, a linear homogeneous function of*

(87) $$J_4,\ \pi_2,\ \pi_2 J_4,\ P^e,\ P^e F_4 \qquad (e = 0, 1, \cdots, 2^n - 1).$$

The values of these invariants for the various classes are here shown.

	$K_{4,0,P}$	$K_{4,1,P}$	C_3	$C_{2,0}$	$C_{2,1}$	C_1	C_0
F_4	0	1	1	0	0	0	0
J_4	0	0	0	0	1	0	1
P	P	P	0	0	0	0	0
π_2	0	0	0	0	0	1	1

* Another proof of (83_1) results from the fact that $P^\nu F_4$ has the value 1 for class $C_{4,1}$ and the value 0 for the remaining classes.

† Another proof follows from the fact that the second member of (84) has the value 1 for classes $C_{2,1}$ and $C_{2,0}$, the value 0 for the remaining classes.

‡ For $n = 1$, Proceedings of the London Mathematical Society, vol. 5 (1907), pp. 303–311. The invariants A_4, I_4 are the π_2, I_0 of the present paper. The explicit expressions for J_4 and F_4 are given on p. 308 and p. 310.

To prove (end of § 5) the independence of F_4, we employ C_3 and $C_{2,0}$; for J_4, C_1 and C_0; for P, $K_{4,1,p}$ and C_3; for π_2, $C_{2,0}$ and C_1. *Hence the four invariants are independent.* It follows from the table that

$$(88)\quad F_4J_4=F_4\pi_2=PJ_4=P\pi_2=0, \qquad P^{2^n}=P, \qquad I^2=I\,(I=F_4, J_4 \text{ or } \pi_2).$$

Any product of two invariants (87) *reduces to one of the set by means of* (88).

Reduction of binary cubic forms in the $GF[p^n]$.

21. If $p^n=3l+2$ or 3^n, every element $\neq 0$ of the $GF[p^n]$ is a cube:

$$e=e^{-3l} \qquad \text{or} \qquad e^{3\cdot 3^{n-1}},$$

respectively; hence every element has an unique cube root in the field. But if $p^n=3l+1$, just one-third of the elements $\neq 0$ are cubes. If ϵ is a primitive root in the field, the cubes are $\epsilon^{3i}\,(i=1,\cdots,l)$; while the not-cubes are the products of the preceding by ϵ, ϵ^2. We shall set

$$(89)\quad \beta=1 \text{ if } p^n=3l+2 \text{ or } p^n=3^n; \qquad \beta=1,\ \epsilon \text{ or } \epsilon^2 \text{ if } p^n=3l+1.$$

Consider the binary cubic with coefficients in the $GF[p^n]$,

$$(90)\qquad f(x,y)\equiv a_0x^3+a_1x^2y+a_2xy^2+a_3y^3.$$

If ρ_1, ρ_2, ρ_3 are the roots of $f(x,1)=0$, the discriminant of (90) is

$$(91)\quad D=a_0^4\Pi(\rho_i-\rho_j)^2=18a_0a_1a_2a_3-4a_0a_2^3-4a_1^3a_3+a_1^2a_2^2-27a_0^2a_3^2.$$

When $p^n=2$ the special form $f_1=xy(x+y)$ is unaltered under every linear transformation in the field; it is a special case of the canonical form (93). Except when $f\equiv 0$, or when $p^n=2$ and $f=f_1$, we can transform f into a form f' with $a_0'\neq 0$. If $a_0=0$, $a_3\neq 0$, we apply $(y,-x)$. If $a_0=a_3=0$, we apply $(x,\lambda x+y)$ and have $a_0'=a_1\lambda+a_2\lambda^2$.

In the normalization of a form f with $a_0\neq 0$, we consider several cases.

(i) In case there is a triple root ρ, we apply the transformation

$$x-\rho y=\alpha x', \qquad y=\alpha^{-1}y',$$

of determinant unity. We obtain $a_0\alpha^3x'^3$, and hence βx^3.

(ii) In case there is a double root ρ_1 and a simple root ρ_2, we apply the transformation of determinant unity

$$(92)\qquad x-\rho_1y=\alpha x', \qquad x-\rho_2y=\alpha^{-1}(\rho_1-\rho_2)y'.$$

We obtain cx'^2y', where $c=a_0\alpha(\rho_1-\rho_2)$ may be made unity by choice of α.

(iii) Let $f(x,1)=0$ have three distinct roots ρ_i in the field. Applying the

transformation (92) to $f(x, y)$, we obtain $kx'y'(x' + ry')$, where

$$k = a_0\alpha(\rho_3 - \rho_2), \qquad r = \alpha^{-2}(\rho_1 - \rho_2)(\rho_3 - \rho_1)/(\rho_2 - \rho_3).$$

We make $k = 1$ by choice of α. Then $r^2 = D$, by (91). In

$$xy(x + ry), \qquad r^2 = D, \tag{93}$$

we may change the sign of r by applying $(ry, -r^{-1}x)$. Under the group G_1 of binary transformations of determinant unity, the canonical form is (93), where r is a particular square root of D.

Under the total group G, a canonical form is

$$\beta xy(x + y) \quad \text{or} \quad xy(x + \beta y). \tag{93'}$$

Indeed, $(sx, r^{-1}sy)$ replaces (93) by $cxy(x + y)$, $c = r^{-1}s^3$.

(iv) Let f be the product of an irreducible quadratic factor and a linear factor. The latter may be taken to be a multiple of x.

If $p \neq 2$, we apply $(x, y + ax)$ and obtain

$$dx(y^2 - \mu x^2), \qquad \mu \text{ a not-square.}$$

Applying $(dx, d^{-1}y)$, we obtain the canonical form under G_1

$$x(y^2 - \sigma x^2), \qquad \sigma \text{ a not-square, } \sigma = \tfrac{1}{4}D. \tag{94}$$

As the canonical form under the total group G, we may take *

$$x(y^2 - \nu\beta^2 x^2), \qquad \nu \text{ a particular not-square.} \tag{94'}$$

The latter is obtained from (94) with $\sigma = \nu t^2$ by applying $(s^{-2}x, sy)$, where s is chosen to make ts^{-3} take one of the values β.

If $p = 2$, $f = dx(xy + ax^2 + by^2)$, $dab \neq 0$. Applying $(b^{\frac{1}{2}}x, b^{-\frac{1}{2}}y)$, we obtain a form $mx(xy + y^2 + \delta x^2)$. Applying $(x, y + tx)$, we find that δ is replaced by $c = \delta + t + t^2$. This equation is solvable for t in the $GF[2^n]$ if and only if $\chi(c) = \chi(\delta)$, where

$$\chi(c) = \sum_{i=0}^{n-1} c^{2^i}, \qquad \chi^2 \equiv \chi, \qquad \chi \equiv 0 \text{ or } 1. \tag{95}$$

If $\chi(\delta) = 0$, the quadratic factor would be reducible. Hence

$$mx(xy + y^2 + cx^2), \quad c \text{ a particular root of } \chi(c) = 1, \tag{96}$$

are the canonical forms under G_1. If we multiply x and y by a suitably chosen element, we obtain as the canonical form under the total group G

$$\beta x(xy + y^2 + cx^2). \tag{96'}$$

* We may take $\beta x(x^2 - \nu y^2)$, obtained from (94) with $\sigma = \nu t^2$ by applying $(ax, \nu a t y)$, where a is chosen to make $-a^2\nu t^2 = \beta$.

(v) Finally, let f be irreducible in the $GF[p^n]$. Then $f(x, 1) = 0$ has a root ρ in the $GF[p^{3n}]$, the remaining roots being ρ^{p^n}, $\rho^{p^{2n}}$. For

$$x = \alpha x' + \beta y', \qquad y = \gamma x' + \delta y' \qquad (\Delta = \alpha\delta - \beta\gamma \neq 0), \tag{97}$$

we have

$$x - \rho y = (\alpha - \rho\gamma)(x' - \lambda y'), \qquad \lambda \equiv \frac{\delta\rho - \beta}{-\gamma\rho + \alpha}. \tag{98}$$

so that ρ is transformed by the inverse of the linear fractional form of (97). Hence when $\alpha, \beta, \gamma, \delta$ take all values in the $GF[p^n]$ for which $\Delta \neq 0$, the fraction λ takes $p^n(p^{2n} - 1)$ distinct values in the $GF[p^{3n}]$, no one belonging to the $GF[p^n]$. Hence λ can be made equal to any assigned element of the $GF[p^{3n}]$, not occurring in the $GF[p^n]$; the ratios of the coefficients in (97) are thereby uniquely determined. The canonical forms under the total group G of all transformations (97) are therefore βC, where C is a particular irreducible cubic $x^3 + \cdots$. We may take

$$C = x^3 - xy^2 + \tau y^3,$$

where τ is suitably chosen. For, if $z^3 - z + \tau = 0$ were reducible for every τ in the $GF[p^n]$, $z^3 - z$ would take p^n distinct values when z does, whereas $z^3 - z$ vanishes for $z = 0, \pm 1$.

It remains to discuss the normalization of f under the group G_1 of the $(p^{2n} - 1)p^n$ transformations (97) with $\Delta = 1$. Then, in (98), $\lambda = \rho$ only when

$$\alpha = \delta = \pm 1, \qquad \beta = \gamma = 0.$$

For $p = 2$, the corresponding transformation (97) is the identity, and λ takes $p^n(p^{2n} - 1)$ distinct values; the $2^n - 1$ canonical forms are therefore mC. Henceforth, let $p > 2$. Then λ takes only $\frac{1}{2}p^n(p^{2n} - 1)$ values. Hence the roots of the various irreducible cubic equations fall into two types. Since $\Delta = 1$,

$$\lambda = -\delta\gamma^{-1} + \gamma^{-1}(\alpha - \gamma\rho)^{-1}, \qquad \lambda - \lambda^{p^n} = (\rho - \rho^{p^n})(\alpha - \gamma\rho)^{-1}(\alpha - \gamma\rho^{p^n})^{-1},$$

when $\gamma \neq 0$. But the second equation is true also when $\gamma = 0$. Thus

$$\phi(\lambda) = \phi(\rho) \div a_0^{-2} f^2(\alpha, \gamma), \qquad \phi(\rho) \equiv (\rho - \rho^{p^n})(\rho^{p^n} - \rho^{p^{2n}})(\rho^{p^{2n}} - \rho). \tag{99}$$

Since $\phi^{p^n} = \phi$, $\phi(\rho)$ is an element of the $GF[p^n]$; in particular, it is unaltered when ρ is replaced by another root ρ^{p^n} or $\rho^{p^{2n}}$ of the same cubic. Hence if ρ and λ are roots of two irreducible cubic equations such that λ is a linear fractional function of ρ of determinant a square (which may evidently be made unity), the ratio of $\phi(\lambda)$ to $\phi(\rho)$ is a non-vanishing square in the field. But if this determinant is a not-square, the ratio of the ϕ's is a not-square. It suffices to prove the latter for a particular linear fractional transformation of determinant a not-square ν, for example, $\lambda = \nu\rho$, whence $\phi(\lambda) = \nu^3\phi(\rho)$. Further, it was

shown above that there exists a linear fractional transformation replacing ρ by any root of any irreducible cubic. Hence if ρ and λ are roots of the equations corresponding to two irreducible cubic forms C and C', then C is equivalent to a multiple of C' under the group G_1 if and only if the ratio of $\phi(\lambda)$ to $\phi(\rho)$ is a square.

Since there are $\frac{1}{6}(p^{2n}-1)p^n$ distinct cubic equations of each type, there are exactly 6 binary transformations of determinant unity which multiply a given C by a constant, necessarily ± 1 by (99_1).

Under G_1 the non-equivalent canonical forms may be taken to be mC_1 and mC_2, where C_1 and C_2 are particular irreducible cubic forms $x^3+\cdots$, such that the equation corresponding to C_1 has a root ρ with $\phi(\rho)=1$, that corresponding to C_2 a root ρ with $\phi(\rho)=\nu$, where ν is a fixed not-square. Further, m takes only one of each pair of non-vanishing values $\pm M$. A cubic form, for which the invariant E (§ 22) is not zero, can be transformed into mC_1 or mC_2 according as $E=m^2$ or $m^2\nu$.

The invariants of the binary cubic form in the $GF[p^n]$, § 22–26.

22. The results under case (v) of § 21 lead us quite naturally to an important invariant E of the binary cubic $f(x, y)$ under the group G_1 of transformations of determinant unity. According to our general standpoint, there exists an invariant which takes prescribed values for each class under G_1. Let E be zero for all reducible cubics, $E=m^2$ for the cubic mC_1, $E=m^2\nu$ for mC_2. Thus E has the value $E_i=m^2\phi(\rho)$ for mC_i ($i=1$ or 2). We readily deduce the value of E for any irreducible cubic $f(x, y)$. One of the cubics mC_i can be transformed with G_1 into f. If λ is a root of $f(x, 1)=0$, (99_1) gives $\phi(\lambda)=\phi(\rho)m^2/a_0^2$, since the coefficient a_0 of $f(x, y)$ is the value of mC_i for $x=\alpha$, $y=\gamma$. Hence $a_0^2\phi(\lambda)=E_i$. Thus, for any irreducible cubic $f(x, y)$, E has the value $a_0^2\phi(\lambda)$. For a reducible cubic, $E=0$ by definition. Hence* for an arbitrary cubic $f(x, y)$, $E=a_0^2R$, where R is the resultant of $x^{p^n}=x$ and $f(x, 1)=0$, the constant factor being determined so that

$$R=\prod_{i=1}^{3}(x_i-x_i^{p^n}), \tag{100}$$

where the x_i are the roots $f(x, 1)=0$. Under the transformation $(x, \delta y)$, each root is multiplied by the determinant δ; then R is multiplied by δ^3, while a_0 is unaltered. *Hence E is a relative invariant of weight* 3.

23. We seek absolute characteristic invariants of the d classes represented by βx^3, where d is the greatest common divisor of $\mu=p^n-1$ and 3, and β is de-

* For a different, but equivalent, definition of E, see these Transactions, loc. cit., p. 307. For the explicit expressions for E for various values of p^n, see pp. 208, 212, 229, 230, 231.

fined by (89). To this end we construct a polynomial Q in the a_i which has the value $\beta^{\mu/d}$ for the class βx^3 (for each of the d values of β) and the value zero for all classes other than these d; then Q will be an absolute invariant of the cubic form. If $d=1$, Q itself is the desired characteristic invariant. If $d=3$, $\omega=\epsilon^{\mu/3}$ is a cube root of unity in the field, and

$$(101)\quad \tfrac{1}{3}(Q+Q^2+Q^3),\qquad \tfrac{1}{3}(\omega^2 Q+\omega Q^2+Q^3),\qquad \tfrac{1}{3}(\omega Q+\omega^2 Q^2+Q^3)$$

are characteristic absolute invariants for the classes x^3, ϵx^3, $\epsilon^2 x^3$, respectively.

The cubic form $f(x, y)$, given by (90), has a triple root if and only if

$$(102)\quad A=a_1^2-3a_0a_2,\qquad B=a_2^2-3a_1a_3,\qquad C=a_1a_2-9a_0a_3$$

all vanish in the field, a result valid for any p. Let

$$(103)\qquad \pi=(1-A^\mu)(1-B^\mu)(1-C^\mu).$$

Evidently $Q\equiv\pi q$. To determine q, we consider the sets a_i for which A, B, C all vanish, so that $Q=q$. For the sets with $a_0\neq 0$, f can be transformed into a_0x^3, so that $Q=a_0^{\mu/d}$. Hence

$$Q\equiv a_0^{\mu/d}+c(1-a_0^\mu),$$

for all the sets. For the sets with $a_0=0$, $a_3\neq 0$, $f\equiv a_3y^3$, so that

$$Q=a_3^{\mu/d}=c,\qquad c\equiv a_3^{\mu/d}+k(1-a_3^\mu).$$

Then for the sets with $a_0=a_3=0$, $f\equiv 0$, $Q=0$, whence $k=0$. Thus

$$(104)\qquad Q=\pi\,[\,a_0^{\mu/d}+a_3^{\mu/d}(1-a_0^\mu)\,].$$

For $d=1$, the second factor of Q equals $1-t$, where

$$t=(a_0^\mu-1)(a_3^\mu-1).$$

Since $a(a^\mu-1)=0$, we have $\pi t=I_0$,

$$(105)\qquad I_0=\prod_{i=0}^{3}(a_i^\mu-1),\qquad P=(a_1^\mu-1)(a_2^\mu-1),$$

$$(106)\qquad Q=\pi-I_0 \qquad \text{(if } d=1\text{)}.$$

In particular, if $p=3$ (whence $d=1$), $\pi=P$.

For $d=3$, the cube of the final factor in (104) is $1-t$. Hence

$$(107)\qquad Q^3\equiv\pi-I_0,\qquad Q^4=Q \qquad \text{(for any } d\text{)}.$$

24. Let J be the characteristic absolute invariant of the class represented by x^2y. Then $J=(1-D^\mu)L$. Consider the sets a_i with $D=0$, $A\neq 0$, and then the sets with $D=A=0$, $B\neq 0$. We find that

$$L=1+m(A^\mu-1)(B^\mu-1).$$

Consider the sets with $D=A=B=0$. Then $C=0$, since

$$-3D\equiv C^2-4AB, \tag{108}$$

so that $0=1+m$. Hence

$$J=(1-D^\mu)\{1-(1-A^\mu)(1-B^\mu)\}. \tag{109}$$

We may give J a more symmetrical form. If $p\neq 3$,

$$D^\mu=(C^2-4AB)^\mu,$$

by (108). Also $A(A^\mu-1)=0$. Hence, by (103),

$$J=1-D^\mu-\pi. \tag{110}$$

This result is true also if $p=3$, since then $DP=0$, $\pi=P$.

In view of (110), π is an absolute invariant of the cubic.

25. We can now prove that every invariant of the binary form in the $GF[p^n]$, $p>2$, is a rational integral function of D, E, Q, I_0. Under the group G_1 of transformations of determinant unity, a complete set of non-equivalent classes are defined by the representative forms $f\equiv 0$, βx^3, x^2y, (93), (94), mC_1, mC_2, in which r and m each take $\frac{1}{2}\mu$ values $\neq 0$, no one value being the negative of another. Here $\mu=p^n-1$. Hence, for $p>2$, the number of classes is

$$1+d+1+\tfrac{1}{2}\mu+\tfrac{1}{2}\mu+\tfrac{1}{2}\mu+\tfrac{1}{2}\mu=d+2+2\mu.$$

Hence by § 4 there are exactly $d+2+2\mu$ linearly independent invariants under G_1. These may be taken to be

$$I_0\ Q^\delta\ (\delta=1,\cdots,d),\ J,\ D^i,\ E^i\ (i=1,\cdots,\mu). \tag{111}$$

We note that I_0 and J are characteristic invariants of the classes $f\equiv 0$, x^2y; likewise Q for x^3 if $d=1$, and the linear combinations (101) for the βx^3 if $d=3$. Suitable linear combinations of the E^i give the characteristic invariants of the classes mC_j, while linear combinations of the $(E^\mu-1)D^i$ give those for the classes (93) and (94). For an irreducible cubic, $D=E^2$ by (91) and (100). Hence for every cubic,

$$DE=E^3,\qquad D^iE^k=E^{k+2i},\qquad (E^\mu-1)D^i=E^{2i}-D^i.$$

Hence, as in § 4, the invariants (111) are linearly independent.* Of these,

* If N is the number of sets of solutions $x:y$ of $f(x,y)=0$ in the $GF[p^n]$, there exists an absolute invariant K for which $K\equiv N-1 \pmod p$, with $K=0$ if f is identically zero, Bulletin of the American Mathematical Society, vol. 14 (1908), p. 316. For $p>2$, we may give to K the compact expression

$$K=J-E^\mu+(1-E^\mu)(D^\mu+D^{\frac{1}{2}\mu}).$$

I_0, Q and J are absolute invariants, while the powers of D and E are relative invariants under the total group G.

THEOREM. *The $d + 2p^n$ invariants* (111) *give a complete set of linearly independent invariants of the binary cubic form in the $GF[p^n]$, $p > 2$.*

We may suppress J and introduce 1, in view of (107), (110). We may suppress Q^i and introduce π, etc. Any product of D, E, π, Q, I_0 can be reduced to a linear homogeneous function of the invariants (111), by means of the relations (valid for any p):

$$\begin{aligned} &DE = E^3, \qquad D\pi = DQ = DI_0 = E\pi = EQ = EI_0 = QI_0 = 0, \\ &\pi Q = Q, \quad \pi I_0 = I_0, \quad I_0^2 = I_0, \quad Q^4 = Q, \quad D^{\mu+1} = D, \quad E^{\mu+1} = E. \end{aligned} \tag{112}$$

Additional invariants of a cubic in the $GF[2^n]$.

26. For $p = 2$, there are $d + 2 + 3\mu$ classes represented by $f \equiv 0$, βx^3, $x^2 y$, (93), (96), and in mC, with $\mu = 2^n - 1$ forms in each of the last three sets. Hence we require μ invariants in addition to (111). These additional invariants together with the powers of D should enable us to differentiate the 2μ classes represented by (93) and (96), and hence to distinguish between the types of reducible cubics having no double root.

We seek the necessary and sufficient conditions that $f(x, y)$ shall be the product of a linear and an irreducible quadratic factor in the $GF[2^n]$. If $a_0 = 0$, $a_1 x^2 + a_2 xy + a_3 y^2$ must be irreducible, so that, by (42), (55),

$$\chi(a_1 a_3 a_2^{2^n-3}) = 1. \tag{113}$$

Next, let $a_0 \neq 0$. Let $a_0 x + a_1 y = \xi$, $y = \eta$. Then

$$a_0^2 f(x, y) = \xi^3 + e\xi\eta^2 + g\eta^3, \qquad e = a_1^2 + a_0 a_2, \qquad g = a_0 D^{\frac{1}{2}}. \tag{114}$$

If $e = 0$, the conditions require that there exist one and but one cube root of g; hence there must be a single root of $\omega^3 = 1$, so that $2^n - 1$ must be prime to 3. When the latter condition is satisfied, every element is a cube (§ 21). Hence if $e = 0$, the desired necessary and sufficient conditions are that n be odd and that $g \neq 0$. Finally, let $e \neq 0$. Let $X = e^{-\frac{1}{2}}\xi$, $Y = \eta$. Then

$$e^{-\frac{3}{2}} a_0^2 f(x, y) = X^3 + XY^2 + tY^3 \qquad (t = ge^{-\frac{3}{2}}). \tag{115}$$

If $t = 0$, we obtain $X(X + Y)^2$. Hence must $t \neq 0$. Our problem thus reduces to the characterization of the values $\neq 0$ of t for which there is one and but one root in the $GF[2^n]$ of the equation

$$z^3 = z + t. \tag{116}$$

We shall investigate the relations between the α's and β's in

$$z^{2^k} - z = \alpha_k^2 z^2 + \beta_k z. \tag{117}$$

We have the initial sets of values

$$(118)\qquad \alpha_1=\beta_1=1,\qquad \alpha_2=1,\qquad \beta_2=t+1,\qquad \alpha_3=\beta_3=t+1.$$

Squaring (117) and eliminating z^4, we find that

$$(119)\qquad \alpha_{k+1}=\alpha_k^2+\beta_k+1,\qquad \beta_{k+1}=t\alpha_k^4+1.$$

As k increases, the expressions for the α_k, β_k in terms of t increase rapidly in complexity. We shall prove by induction that

$$(120)\qquad \beta_k=\alpha_k^2+t\alpha_k\alpha_{k-1}^2.$$

The expression for β_{k+1} analogous to (120) will equal (119_2) if

$$t\alpha_k^2[(t\alpha_{k-1}^4+1)+(\alpha_k^2+t\alpha_k\alpha_{k-1}^2)]=0,$$

as seen by eliminating α_{k+1} by means of (119). But the quantity in brackets reduces to $\beta_k+\beta_k\equiv 0$ by (119) with k replaced by $k-1$, and (120). In view of (118), relation (120) holds for $k=2$ and $k=3$. Hence the induction is complete. Thus

$$(121)\qquad \alpha_{k+1}=1+t\alpha_k\alpha_{k-1}^2,\qquad \alpha_{k+1}=\alpha_k^2+t\alpha_{k-1}^4.$$

Upon equating the second members, we obtain an equation designated (121′).

We next prove the following relation between two α's:

$$(122)\qquad \alpha_{k+1}+t\alpha_{k+1}^2\alpha_k^2+\alpha_k^2=t^{2^k-1}.$$

It holds for $k=1$ and $k=2$ by (118). Assume it true as far as $k-1$. Then

$$\alpha_k+t\alpha_k^2\alpha_{k-1}^2+\alpha_{k-1}^2=t^{2^{k-1}-1}.$$

Multiplying the square of the latter by t, we see that (122) is true if

$$(\alpha_{k+1}+\alpha_k^2+t\alpha_{k-1}^4)+t\alpha_k^2(\alpha_{k+1}^2+1+t^2\alpha_k^2\alpha_{k-1}^4)=0.$$

The first part vanishes by (121_2), the second by the square of (121_1).

By (117) for $k=n$, a root of (116) belongs to the $GF[2^n]$ if and only if it satisfies also $\alpha_n^2z+\beta_n=0$; in fact, $z\neq 0$ since $t\neq 0$. If $\alpha_n=0$, then $\beta_n=0$ by (120) for $k=n$, and the cubic has three distinct roots in the field. Hence $\alpha_n\neq 0$ is a necessary condition that the cubic shall have a single root in the field. By (120),

$$(123)\qquad z=1+t\alpha_{n-1}^2\alpha_n^{-1}.$$

This uniquely determined value is actually a root of (116) if and only if $R_n=0$, where

$$(124)\qquad R_k=\alpha_k^3+t\alpha_k\alpha_{k-1}^4+t^2\alpha_{k-1}^6.$$

Multiply (121′) by α_k, (121_2) by $t\alpha_{k-1}^2$, and add. Thus

$$(124')\qquad R_k=\alpha_k+t\alpha_{k+1}\alpha_{k-1}^2.$$

In R_{k+1}, obtained from (124′), we replace α_{k+2} by its value from (121_1). Then

$$R_k^2 + R_{k+1} = (1+t)\alpha_k^2 + \alpha_{k+1} S, \qquad S = 1 + t^2\alpha_k^4 + t^2\alpha_{k+1}\alpha_{k-1}^4.$$

In S we replace α_{k+1} by its value (121_2), and then add the square of (121′):

$$S = (1+t)\{t\alpha_k^4 + (\alpha_k^4 + t^2\alpha_{k-1}^8)\}.$$

The quantity in the last parenthesis equals α_{k+1}^2 by (121_2). Hence

$$R_k^2 + R_{k+1} = (1+t)(\alpha_k^2 + t\alpha_k^4\alpha_{k+1} + \alpha_{k+1}^3).$$

To the last factor add the product of α_{k+1} by (121′) with k replaced by $k+1$; there results the left member of (122). Hence

$$R_k^2 + R_{k+1} = t^{2^k} + t^{2^k-1}. \tag{125}$$

Raise to the power 2^i the equation (125) for $k = n-i-1$. Then

$$R_{n-i}^{2^i} + R_{n-i-1}^{2^{i+1}} = t^{2^{n-1}} + t^{2^{n-1}-2^i}.$$

Forming the sum of the latter for $i = 0, 1, \cdots, n-3$, we get

$$R_n = (n-1)t^{2^{n-1}} + \sum_{i=0}^{n-1} t^{2^{n-1}-2^i}, \tag{126}$$

upon replacing R_2 by its value $t^2 + t + 1$. Hence, finally,

$$R_n = t^{2^{n-1}}\{n - 1 + \chi(t^{-1})\}, \tag{127}$$

where χ is the function (42). We note that for $R_n = 0$, $t \neq 0$, then α_n can not vanish; for, if so, (124), with $k = n$, would give $\alpha_{n-1} = 0$, in contradiction with (121_1) for $k = n-1$. Hence we have the

THEOREM. *The cubic $z^3 = z + t$, with $t \neq 0$, has one and but one root in the $GF[2^n]$ if and only if $\chi(t^{-1}) = n - 1$.*

The unique root $z = \rho$ is given by (123), in which the α's are defined by the recursion formulae (121) with the initial values (118). Removing the factor $z - \rho$ from (116), we obtain $z^2 + \rho z + \rho^2 + 1$. Since the latter is irreducible, $\chi(1 + \rho^{-2}) = 1$, whence $\chi(\rho^{-1}) = n - 1$. *When t ranges over the elements $\neq 0$ of the $GF[2^n]$ for which $z^3 = z + t$ has an unique root ρ_t, the latter ranges over the same elements; the function $z^3 - z$ represents a substitution on these elements.*

27. The condition on t becomes $\chi(1 + t^{-2}) = 1$ upon applying

$$\chi(1+s) = n + \chi(s), \qquad \chi(s) = \chi(s^2).$$

We insert the value of t from (115). Hence the condition is

$$\chi(1 + g^{-2}e^3) = 1$$

if $eg \neq 0$. For $e = 0$, $g \neq 0$, this is equivalent to the condition, obtained

above for this case, that n is odd. Hence for $g \neq 0$, e arbitrary, the condition is

$$\chi(\epsilon) = 1, \qquad \epsilon = (g^2 + e^3) g^{2^n - 3}, \tag{128}$$

where (and below) $2^n - 3$ is to be replaced by unity if $n = 1$.

Let H be the characteristic absolute invariant for the class of cubic forms having a linear and an irreducible quadratic factor, so that $H = 1$ for such a cubic, while $H = 0$ for all others. If $a_0 \neq 0$, we have $H = \chi(\epsilon)$, ϵ being defined by (128). For, if $g = 0$, then $D = 0$, so that $H = 0$ by definition. Hence in every case

$$H = \chi(\epsilon) + m(a_0^\mu - 1) \qquad (\mu = 2^n - 1).$$

Let $a_0 = 0$; then m equals the left member of (113). Without altering H, we may add $\chi(a_0 a_3^2 a_2^{2^n - 4})$ to m. Inserting the values (114) of e, g, we get

$$H = \chi(\lambda), \quad \lambda = \Delta\{(a_1^6 + a_1^4 a_0 a_2 + a_0^3 a_2^3 + a_0^4 a_3^2) a_0 a_0^\nu \Delta^\nu + a_3 a_2^\nu (a_0^\mu - 1)\}, \tag{129}$$

where $\mu = 2^n - 1$, $\nu = 2^n - 4$, the latter being replaced by zero if $n = 1$: while

$$\Delta = a_0 a_3 + a_1 a_2 = D^1 = D^{2^{n-1}}.$$

For $n = 1$ and $n = 2$, λ becomes

$$a_0 a_3 + a_0 a_1 a_2 + a_0 a_1 a_3 + a_0 a_2 a_3 + a_1 a_2 a_3 + a_0 a_1 a_2 a_3,$$

$$a_0 a_1 a_2 + a_0 a_1^3 a_3^2 + a_0 a_2^3 a_3^2 + a_1 a_2 a_3 + a_0 a_1^2 a_2^2 a_3,$$

and $H = \Delta(1 + R + K)$, ΔR, respectively, where R and K are invariants given in these Transactions, vol. 8 (1907), p. 222, p. 230.

For general n, linear combinations of the $D^i H$ $(i = 0, \cdots, \mu - 1)$ give characteristic invariants under G_1 for the classes defined by (96), whose discriminant D is m^4.

THEOREM. *The $d + 2 + 3\mu$ invariants* (111) *and $D^i H$ $(i = 0, \cdots, \mu - 1)$ give a complete set of linearly independent invariants of the binary cubic form in the $GF[2^n]$.*

The product of any two of these invariants can be reduced to a linear combination of them by means of relations (112) and

$$H^2 = H, \qquad D^\mu H = H, \qquad H I_0 = HQ = HJ = H\pi = HE = 0. \tag{130}$$

THE UNIVERSITY OF CHICAGO,
December 1, 1908.

BEITRÄGE ZUR THEORIE DER GRUPPEN LINEARER HOMOGENER SUBSTITUTIONEN*

VON

I. SCHUR

In seiner Arbeit *Über die vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören* † hat Herr A. LOEWY eine wichtige neue Art der Reduktion einer Gruppe $\mathfrak{G}$ linearer homogener Substitutionen angegeben, die er als die Zerlegung der Gruppe unter Hervorhebung der zu ihr gehörenden auf einander folgenden grössten vollständig reduziblen Gruppen bezeichnet. Diese Zerlegung kann man, wenn die Koeffizienten aller Substitutionen von $\mathfrak{G}$ einem gegebenen Zahlkörper Ω angehören, auch innerhalb des Körpers Ω durchführen. Auf diese Weise erhält Herr LOEWY ein System von gewissen zu $\mathfrak{G}$ gehörenden Gruppen $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_\mu$ mit Koeffizienten aus dem Körper Ω, die inbezug auf Ω vollständig reduzibel sind, und er zeigt auch, dass die Gruppen $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_\mu$, wenn man äquivalente (ähnliche) Gruppen als nicht von einander verschieden ansieht, in der gegebenen Reihenfolge als eindeutig bestimmt zu betrachten sind.

In der vorliegenden Arbeit soll nun eine, wie ich glaube, nicht unwesentliche Ergänzung der LOEWY'schen Resultate mitgeteilt werden. Es wird nämlich gezeigt, dass die Gruppen $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_\mu$, wenn wieder äquivalente Gruppen als nicht von einander verschieden gelten, von der Wahl des Körpers Ω gänzlich unabhängig sind; oder genauer: legt man der Betrachtung an Stelle des Körpers Ω einen anderen Körper Ω' zugrunde, der ebenfalls alle Substitutionskoeffizienten von $\mathfrak{G}$ umfasst, und sind $\mathfrak{A}'_1, \mathfrak{A}'_2, \cdots, \mathfrak{A}'_{\mu'}$ die zu $\mathfrak{G}$ gehörenden auf einander folgenden grössten vollständig reduziblen Gruppen inbezug auf Ω', so ist $\mu' = \mu$, ferner sind $\mathfrak{A}_\alpha$ und $\mathfrak{A}'_\alpha$ äquivalente Gruppen.‡

Zur Begründung dieses Resultats bedarf es einer eingehenderen Untersuchung der in einem Körper Ω irreduziblen Gruppen linearer homogener Substitutionen. In §§ 2–4 stelle ich für diese Gruppen eine Reihe von Sätzen auf, die

* Presented to the Society (Chicago) April 17, 1908.

† Transactions of the American Mathematical Society, vol. 6 (1905), p. 504. Vergl. auch L. STICKELBERGER, *Zur Theorie der vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören*, ibid., vol. 7 (1906), p. 509.

‡ Für den Fall, dass $\mathfrak{G}$ eine zyklische Gruppe ist, die aus den Potenzen einer Substitution besteht, hat diesen Satz bereits Herr J. WIRTH in seiner Dissertation *Über die Elementarteiler einer linearen homogenen Substitution* (Freiburg i. Br., 1906) bewiesen.

ganz analog sind den Sätzen, die ich für den speziellen Fall der endlichen Gruppen bereits in meiner Arbeit *Arithmetische Untersuchungen über endliche Gruppen linearer Substitutionen* * bewiesen habe. Dass diese Satze auch für unendliche Gruppen ihre Geltung behalten, beruht in erster Linie auf dem Umstand, dass jede Gruppe linearer Substitutionen, die inbezug auf einen gegebenen Körper Ω irreduzibel ist, im Bereich aller Zahlen vollständig reduzibel ist.† Auf diese bemerkenswerte Tatsache hat bereits Herr TABER am Schluss seiner Arbeit *Sur les groupes réductibles de transformations linéaires et homogènes* ‡ aufmerksam gemacht. Nimmt man diesen Satz als bekannt an, so lassen sich auch meine anderen Sätze, insbesondere der hier hauptsächlich in Betracht kommende Satz X leichter beweisen. Es schien mir aber von Interesse, die Untersuchung in der Weise durchzuführen, dass sich zugleich ein neuer Beweis für den TABER'schen Satz ergiebt. Während Herr TABER beim Beweis seines Satzes von der Theorie der hyperkomplexen Grössen Gebrauch macht, stützt sich meine Beweisführung auf einen einfachen Hilfssatz (Satz II), der sich leicht direkt begründen lässt.

§ 1.

Unter einer Gruppe linearer Substitutionen (Matrizen) des *Grades* g verstehe ich im folgenden ein beliebiges endliches oder auch unendliches System $\mathfrak{G}$ von linearen homogenen Substitutionen in g Variabeln, wenn das Produkt von je zwei Substitutionen von $\mathfrak{G}$ wieder in $\mathfrak{G}$ enthalten ist. Sind die Koeffizienten aller Substitutionen von $\mathfrak{G}$ Zahlen eines gegebenen Zahlkörpers Ω, so sage ich kurz, $\mathfrak{G}$ sei eine *in* Ω *rationale Gruppe*. Ebenso verstehe ich unter einer *in* Ω *rationalen Matrix* eine Matrix, deren Koeffizienten dem Körper Ω angehören. Geht nun eine in Ω rationale Gruppe $\mathfrak{G}$ durch die lineare Transformation T der Variabeln in die ihr äquivalente Gruppe $\mathfrak{G}' = T\mathfrak{G}T^{-1}$ über, und weiss man, dass $\mathfrak{G}'$ wieder in Ω rational ist, so kann man offenbar T auch so wählen, dass die Koeffizienten von T ebenfalls in Ω enthalten sind.

Es seien nun $\mathfrak{G}$ und $\mathfrak{H}$ zwei Gruppen linearer Substitutionen der Grade g und h, die entweder einander isomorph oder allgemeiner einer dritten Gruppe homomorph § sind. Lässt sich dann eine Matrix P mit g Zeilen und h Kolonnen oder auch eine Matrix P mit h Zeilen und g Kolonnen bestimmen, sodass für je zwei entsprechende Substitutionen (Matrizen) G und H der beiden Gruppen $\mathfrak{G}$ und $\mathfrak{H}$ die Gleichung $GP = PH$, bezw. die Gleichung $PG = HP$

*Sitzungsberichte der Berliner Akademie, Februar 1906, p. 164. Im folgenden wird diese Arbeit kurz mit A. zitiert.

†Für den Fall, das Ω den Bereich aller reellen Zahlen bedeutet, hat diesen Satz Herr A. LOEWY in seiner Arbeit *Über die Reducibilität der reellen Gruppen linearer homogener Substitutionen*, Transactions of the American Mathematical Society, vol. 4 (1903), p. 171, bewiesen.

‡Comptes Rendus de l'Académie des Sciences, April 1906, p. 948.

§ Vergl. G. FROBENIUS und I. SCHUR, *Über die Äquivalenz der Gruppen linearer Substitutionen*, Sitzungsberichte der Berliner Akademie, Februar 1906, p. 209.

besteht, so will ich $\mathfrak{G}$ und $\mathfrak{H}$ als *verkettete* Gruppen bezeichnen; ich sage auch kürzer, die Matrix P genüge der Gleichung

$$(1) \qquad \mathfrak{G}P = P\mathfrak{H} \qquad \text{oder} \qquad P\mathfrak{G} = \mathfrak{H}P.$$

Natürlich soll hierbei nicht $P = 0$ sein. — Sind insbesondere die beiden Gruppen $\mathfrak{G}$ und $\mathfrak{H}$ in einem Körper Ω rational und weiss man, dass eine von Null verschiedene Matrix P existiert, die einer der Gleichungen (1) genügt, so kann man offenbar auch eine in Ω rationale Matrix wählen, die dieselbe Bedingung erfüllt. Sind ferner $\mathfrak{G}'$ und $\mathfrak{H}'$ zwei zu $\mathfrak{G}$ und $\mathfrak{H}$ äquivalente Gruppen, von denen bekannt ist, dass sie untereinander verkettet sind, so schliesst man leicht, dass auch $\mathfrak{G}$ und $\mathfrak{H}$ verkettet sind. Dieser Fall tritt insbesondere ein, wenn die Gruppen $\mathfrak{G}'$ und $\mathfrak{H}'$ die spezielle Form

$$\mathfrak{G}' = \begin{pmatrix} \mathfrak{A} & 0 \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}, \qquad \mathfrak{H}' = \begin{pmatrix} \mathfrak{A} & 0 \\ 0 & \mathfrak{E} \end{pmatrix}$$

aufweisen, wo $\mathfrak{A}$ eine gewisse Gruppe des Grades a bedeuten soll. Denn versteht man alsdann unter P' die Matrix

$$P' = (p_{\kappa\lambda}) \qquad (\kappa = 1, 2, \cdots, h\,;\ \lambda = 1, 2, \cdots, g),$$

wo $p_{11}, p_{22}, \cdots, p_{aa}$ gleich 1, dagegen alle übrigen $p_{\kappa\lambda}$ gleich 0 sind, so wird, wie man leicht findet,

$$P'\mathfrak{G}' = \mathfrak{H}'P' = \begin{pmatrix} \mathfrak{A} & 0 \\ 0 & 0 \end{pmatrix}.$$

Daher sind in diesem Fall $\mathfrak{G}'$ und $\mathfrak{H}'$, und folglich auch $\mathfrak{G}$ und $\mathfrak{H}$ verkettete Gruppen.

Es gilt ferner der für das folgende wichtige Satz:

I. *Es seien $\mathfrak{G}$ und $\mathfrak{H}$ zwei in einem Körper Ω rationale Gruppen linearer Substitutionen der Grade g und h, die entweder isomorph oder einer dritten Gruppe homomorph sind, und es sei $\mathfrak{G}$ insbesondere im Körper Ω irreduzibel. Ist dann P eine in Ω rationale Matrix, die der Gleichung $\mathfrak{G}P = P\mathfrak{H}$ oder der Gleichung $P\mathfrak{G} = \mathfrak{H}P$ genügt, so ist entweder $P = 0$ oder es ist P vom Range g. Im letzteren Falle enthält $\mathfrak{H}$ die Gruppe $\mathfrak{G}$ als irreduziblen Bestandteil inbezug auf Ω.*

Der Beweis dieses Satzes ist genau ebenso zu führen, wie der Beweis des Satzes I meiner mit A. zitierten Arbeit.

Aus dem Satze I ergiebt sich sofort:

I*. *Ist $\mathfrak{G}$ eine im Körper Ω irreduzible Gruppe des Grades g und $\mathfrak{H}$ eine in Ω rationale Gruppe des Grades h, die mit $\mathfrak{G}$ verkettet ist, so muss $h \geqq g$ sein. Ist speziell $h = g$, so muss $\mathfrak{H}$ mit $\mathfrak{G}$ äquivalent sein.*

Um die Untersuchung später nicht unterbrechen zu müssen, schicke ich noch folgende Bemerkung voraus.

Es sei

$$\phi(x) = x^r - c_1 x^{r-1} - c_2 x^{r-2} - \cdots - c_r = 0$$

eine Gleichung mit Koeffizienten aus dem Körper Ω, die in Ω irreduzibel ist; die Wurzeln der Gleichung seien $\rho = \rho_0, \rho_1, \rho_2, \cdots, \rho_{r-1}$. Setzt man

$$R = \begin{pmatrix} c_1 & c_2 & \cdots & c_{r-1} & c_r \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} \rho_0 & 0 & \cdots & 0 \\ 0 & \rho_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \rho_{r-1} \end{pmatrix}$$

und

$$Q = \begin{pmatrix} \rho_0^{r-1} & \rho_1^{r-1} & \cdots & \rho_{r-1}^{r-1} \\ \rho_0^{r-2} & \rho_1^{r-2} & \cdots & \rho_{r-1}^{r-2} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

so wird, wie man leicht erkennt,

$$RQ = \begin{pmatrix} \rho_0^r & \rho_1^r & \cdots & \rho_{r-1}^r \\ \rho_0^{r-1} & \rho_1^{r-1} & \cdots & \rho_{r-1}^{r-1} \\ \cdot & \cdot & \cdot & \cdot \\ \rho_0 & \rho_1 & \cdots & \rho_{r-1} \end{pmatrix} = QI$$

Also ist, da die Determinante der Matrix Q nicht 0 ist,

$$Q^{-1}RQ = P. \tag{2}$$

Hieraus folgt sofort, dass R der Gleichung

$$R^r - c_1 R^{r-1} - c_2 R^{r-2} - \cdots - c_r E = 0 \tag{3}$$

genügt, wo E die Einheitsmatrix bedeutet. — Man habe nun eine Gruppe $\mathfrak{F}$ linearer Substitutionen des Grades f, die im Körper $\Omega(\rho)$ rational ist. Es sei

$$F = \{\phi_{\kappa\lambda}(\rho)\}$$

die Koeffizientenmatrix irgend einer Substitution von $\mathfrak{F}$. Ersetzt man in jeder der in Ω rationalen Funktionen $\phi_{\kappa\lambda}(\rho)$ die Zahl ρ durch die zu ρ konjugierten Zahlen $\rho_1, \rho_2, \cdots, \rho_{r-1}$, so mögen die (zu $\mathfrak{F}$ isomorphen) Gruppen $\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_{r-1}$ entstehen. Substituiert man weiter in $\phi_{\kappa\lambda}(\rho)$ für ρ die Matrix R, so bilden auch die Matrizen

$$\overline{F} = \{\phi_{\kappa\lambda}(R)\},$$

deren Grad gleich rf ist, eine mit $\mathfrak{F}$ isomorphe Gruppe $\overline{\mathfrak{F}}$. Dies folgt daraus, dass jede für ρ bestehende Relation mit Koeffizienten aus dem Körper Ω wegen

der Gleichung (3) richtig bleibt, wenn die Zahl ρ durch die Matrix R ersetzt wird.* Ist nun

$$T = \begin{Vmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & Q \end{Vmatrix}$$

die Matrix des Grades rf, die durch f-malige Aneinanderreihung der Matrix Q entsteht, so wird $T^{-1}\overline{\mathfrak{F}}\,T$, wie aus (2) leicht folgt, eine Gruppe, die durch eine einfache Vertauschung der Zeilen und Spalten in die Gruppe

$$\mathfrak{F}' = \begin{Vmatrix} \mathfrak{F} & 0 & \cdots & 0 \\ 0 & \mathfrak{F}_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \mathfrak{F}_{r-1} \end{Vmatrix}$$

übergeht. *Daher ist die Gruppe $\mathfrak{F}'$ mit der im Körper Ω rationalen Gruppe $\overline{\mathfrak{F}}$ äquivalent.*†

§ 2.

Ist $\mathfrak{G}$ eine im Bereiche Z aller Zahlen irreduzible Gruppe linearer Substitutionen, so hat bekanntlich jede Matrix P, die mit allen Substitutionen (Matrizen) von $\mathfrak{G}$ vertauschbar ist,‡ die Form cE, wo c eine Konstante ist. Weiss man aber umgekehrt, dass jede mit $\mathfrak{G}$ vertauschbare Matrix die Form cE hat, so folgt im allgemeinen noch keineswegs, dass $\mathfrak{G}$ im Bereiche Z irreduzibel ist. Es gilt jedoch der Satz:

II. *Ist $\mathfrak{G}$ eine in einem gegebenen Zahlkörper Ω irreduzible Gruppe und weiss man, dass jede mit $\mathfrak{G}$ vertauschbare Matrix die Form cE hat, so ist $\mathfrak{G}$ auch im Bereiche Z aller Zahlen irreduzibel.*

Es sei nämlich $\mathfrak{G}$ in Z reduzibel. Dann kann man bekanntlich, wenn g den Grad von $\mathfrak{G}$ bedeutet, jedenfalls g^2 Konstanten $k_{\alpha\beta}$, die nicht sämmtlich Null sind, bestimmen, sodass für jede Substitution $A = (a_{\alpha\beta})$ von $\mathfrak{G}$ die Gleichung

$$\sum_{\alpha,\,\beta} k_{\beta\alpha} a_{\alpha\beta} = 0 \tag{4}$$

besteht. Ich will nun zeigen, dass unter den über $\mathfrak{G}$ gemachten Voraussetzungen aus dem Bestehen der Relationen (4) auch die Reduzibilität von $\mathfrak{G}$ inbezug auf Ω geschlossen werden kann, was dann auf einen Widerspruch führt.

Der folgende Beweis ist eine fast wörtliche Nachbildung des Beweises, den Herr FROBENIUS und der Verfasser in der bereits zitierten Arbeit für den

* Vergl. A., p. 172.

† Vergl. L. E. DICKSON, *On the reducibility of linear groups*, Transactions of the American Mathematical Society, vol. 4 (1903), p. 434.

‡ Im folgenden sage ich kurz: die Matrix P ist mit der Gruppe $\mathfrak{G}$ vertauschbar.

bekannten BURNSIDE'schen Fundamentalsatz über irreduzible Gruppen angegeben haben. Nur der grösseren Deutlichkeit wegen soll der Beweis hier ausführlich mitgeteilt werden.

Zunächst ist zu beachten, dass die Konstanten $k_{\alpha\beta}$, da sie sich aus linearen Gleichungen mit Koeffizienten aus dem Körper Ω bestimmen, auch als Zahlen des Körpers Ω angenommen werden können. Man denke sich nun alle in Ω rationalen Matrizen $K=(k_{\alpha\beta})$ ins Auge gefasst, deren Koeffizienten $k_{\alpha\beta}$ den Gleichungen (4) genügen. Unter ihnen seien höchstens s linear unabhängig, etwa $K_1, K_2, \cdots, K_s$. Dann lässt sich jede andere solche Matrix K als lineare homogene Verbindung von $K_1, K_2, \cdots, K_s$ mit in Ω rationalen Koeffizienten darstellen. Nun schliesst man aber sofort, dass auch KA dieselbe Eigenschaft besitzt wie K; daher muss sich KA auf die Form $\sum_\sigma r_\sigma K_\sigma$ bringen lassen, wo die r_σ gewisse Zahlen des Körpers Ω bedeuten. Speziell sei

$$K_\rho A=\sum_{\sigma=1}^{s} r_{\rho\sigma} K_\sigma \qquad (\rho=1, 2, \cdots, s).$$

Die Matrizen $R=(r_{\rho\sigma})$ bilden dann eine in Ω rationale Gruppe $\mathfrak{R}$, die mit $\mathfrak{G}$ homomorph ist. Diese Gruppe $\mathfrak{R}$ kann auch inbezug auf Ω reduzibel sein. In diesem Fall lassen sich $t<s$ linear unabhängige lineare Verbindungen $L_1, L_2, \cdots, L_t$ der K_σ mit in Ω rationalen Koeffizienten angeben, sodass

$$L_\rho A=\sum_{\sigma=1}^{t} s_{\rho\sigma} L_\sigma \qquad (\rho=1, 2, \cdots, t) \tag{5}$$

wird und die (in Ω rationalen) Matrizen $S=(s_{\rho\sigma})$ eine inbezug auf Ω irreduzible Gruppe $\mathfrak{S}$ bilden. — Ist aber $\mathfrak{R}$ in Ω irreduzibel, so denken wir uns $t=s$ und $\mathfrak{S}=\mathfrak{R}$ gesetzt. — Bezeichnet man nun, wenn $L_\varrho=(l_{\alpha\beta}^{(\rho)})$ ist, mit P_α die Matrix

$$P_\alpha=\begin{pmatrix} l_{\alpha 1}^{(1)} & l_{\alpha 2}^{(1)} & \cdots & l_{\alpha g}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ l_{\alpha 1}^{(t)} & l_{\alpha 2}^{(t)} & \cdots & l_{\alpha g}^{(t)} \end{pmatrix},$$

so kann man die Gleichungen (5) auch in der Form

$$P_\alpha A=SP_\alpha \tag{6}$$

schreiben. Die P_α sind aber gewiss nicht alle gleich Null, da sonst auch die L_ρ alle Null wären. Folglich sind $\mathfrak{G}$ und $\mathfrak{S}$ verkettete Gruppen, und da sie beide in Ω irreduzibel sind, so müssen sie nach Satz I einander äquivalent sein. Durch passende Wahl von $L_1, L_2, \cdots, L_t$ kann daher auch erreicht werden, dass $S=A$ wird. Die Gleichungen (6) gehen dann über in die Gleichungen $P_\alpha A=AP_\alpha$, d. h. P_α wird mit $\mathfrak{G}$ vertauschbar. Es muss daher auf Grund unserer Annahme über $\mathfrak{G}$ die Matrix P_α die Form $k_\alpha E$ haben; es wird also

$$l_{\alpha\beta}^{(\gamma)}=k_\alpha e_{\gamma\beta} \qquad (\alpha, \beta, \gamma=1, 2, \cdots, g),$$

wo $e_{\gamma\beta}$ gleich 1 oder gleich 0 zu setzen ist, je nachdem $\gamma = \beta$ oder $\gamma \neq \beta$ ist. Es soll doch aber

$$\sum_{\alpha,\beta} \lambda_{\beta\alpha}^{(\gamma)} a_{\alpha\beta} = 0$$

sein; folglich wird

$$\sum_{\beta} a_{\gamma\beta} k_{\beta} = 0.$$

Wären nun die Konstanten k_β, wie das unsere Annahme, dass die L_γ linear unabhängig sein sollen, erfordert, nicht sämtlich 0, so würden diese Gleichungen besagen, dass die Gruppe $\mathfrak{G}$ mit der ihr homomorphen Gruppe, die aus lauter Nullen besteht, verkettet ist. Dies ist aber nach Satz I nicht möglich.

§ 3.

Allgemein gilt der Satz:

III. *Ist $\mathfrak{G}$ eine im Körper Ω irreduzible Gruppe des Grades g und P eine mit $\mathfrak{G}$ vertauschbare Matrix, deren Koeffizienten dem Körper Ω angehören, so muss die charakteristische Determinante $|xE - P|$ der Matrix P Potenz einer in Ω irreduziblen Funktion $\phi(x)$ sein und es besteht die Gleichung $\phi(P) = 0$.*

Dieser Satz ist dem Satz II meiner mit A. zitierten Arbeit völlig analog und ist ebenso wie dort zu beweisen.

Nun ist die Bedingung dafür, dass die Matrix $P = (p_{\alpha\beta})$ mit der Gruppe $\mathfrak{G}$ vertauschbar sein soll, identisch mit einer Reihe von linearen homogenen Gleichungen für die $p_{\alpha\beta}$ mit Koeffizienten aus dem Körper Ω. Denkt man sich daher die allgemeinste mit $\mathfrak{G}$ vertauschbare Matrix P bestimmt, so erscheinen die $p_{\alpha\beta}$ als lineare homogene Verbindungen von gewissen s Parametern $v_1, v_2, \cdots, v_s$ mit in Ω rationalen Koeffizienten. Sind nun unter den charakteristischen Wurzeln dieser allgemeinsten Matrix P genau r unter einander verschieden, so lassen nach einem bekannten Satz der Algebra für die Parameter v_α auch spezielle rationale Werte einsetzen, so dass die so entstehende spezielle Matrix P ebenfalls genau r verschiedene charakteristische Wurzeln besitzt. Da aber P alsdann eine in Ω rationale Matrix wird, so muss nach Satz III

$$|xE - P| = [\phi(x)]^f$$

sein, wo $\phi(x)$ eine in Ω irreduzible Funktion des Grades r bedeutet und $f = g/r$ zu setzen ist. Ist nun $r = 1$, so folgt aus der Gleichung $\phi(P) = 0$, dass P die Form cE hat und man schliesst auch sofort, dass die allgemeinste mit $\mathfrak{G}$ vertauschbare Matrix ebenfalls diese Form haben muss. In diesem Fall ist daher nach Satz II die Gruppe $\mathfrak{G}$ auch im Bereiche Z aller Zahlen irreduzibel. Ist aber $r > 1$ und

$$\phi(x) = (x - \rho)(x - \rho_1) \cdots (x - \rho_{r-1}),$$

so folgt aus der Gleichung $\phi(P)=0$, weil die ρ_α unter einander verschieden sind, in bekannter Weise, dass die Elementarteiler von $|xE-P|$ sämtlich linear sind.* Hieraus schliesst man leicht, dass sich eine Matrix R von nicht verschwindender Determinante mit Koeffizienten, die dem Körper $\Omega(\rho)$ angehören, bestimmen lässt, so dass

$$P'=RPR^{-1}=\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

wird; hierin soll $M=\rho E_f$ sein, wo E_f die Einheitsmatrix des Grades f bedeutet, während unter N eine gewisse in $\Omega(\rho)$ rationale Matrix des Grades $rf-f$ zu verstehen ist, unter deren charakteristischen Wurzeln alsdann nur die Grössen $\rho_1, \rho_2, \cdots, \rho_{r-1}$ (jede f Mal) vorkommen. Man führe nun an Stelle von $\mathfrak{G}$ die ihr äquivalente, in $\Omega(\rho)$ rationale Gruppe

$$\mathfrak{G}'=R\mathfrak{G}R^{-1}$$

ein. Da nun $\mathfrak{G}'$ mit der Matrix P' vertauschbar ist und die Matrizen M und N keine charakteristische Wurzel gemeinsam haben, so ergiebt sich nach einem bekannten Satz, dass $\mathfrak{G}'$ die Form

$$\mathfrak{G}'=\begin{pmatrix} \mathfrak{F} & 0 \\ 0 & \mathfrak{H} \end{pmatrix}$$

haben muss, wo $\mathfrak{F}$ insbesondere eine Gruppe des Grades f ist. Es mögen nun $\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_{r-1}$, $\overline{\mathfrak{F}}$ und $\mathfrak{F}'$ dieselbe Bedeutung haben wie in § 1. Dann sind (vergl. § 1) $\mathfrak{G}'$ and $\mathfrak{F}'$ unter einander verkettete Gruppen. Daher sind auch die ihnen äquivalenten in Ω rationalen Gruppen $\mathfrak{G}$ und $\overline{\mathfrak{F}}$ verkettet; da aber $\mathfrak{G}$ in Ω irreduzibel und $\overline{\mathfrak{F}}$ von gleichem Grade $g=rf$ wie $\mathfrak{G}$ ist, so müssen $\mathfrak{G}$ und $\overline{\mathfrak{F}}$ nach Satz I' äquivalente Gruppen sein. Hieraus folgt aber, dass $\mathfrak{G}$ auch der Gruppe

$$\mathfrak{F}'=\begin{bmatrix} \mathfrak{F} & 0 & \cdots & 0 \\ 0 & \mathfrak{F}_1 & \cdots & 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ 0 & 0 & \cdots & \mathfrak{F}_{r-1} \end{bmatrix}$$

äquivalent ist.

Die Gruppe $\mathfrak{F}$ ist jedenfalls im Körper $\Omega(\rho)$ irreduzibel.† Denn wäre dies nicht der Fall, so würde man sofort schliessen können, dass die aus $\mathfrak{F}$ hervorgehende in Ω rationale Gruppe $\overline{\mathfrak{F}}$ inbezug auf Ω reduzibel ist; dies ist aber nicht möglich, da $\overline{\mathfrak{F}}$ mit $\mathfrak{G}$ äquivalent ist. Es ist aber $\mathfrak{F}$ *auch im Körper Z aller Zahlen irreduzibel.* Denn wäre die im Körper $\Omega(\rho)$ irreduzible Gruppe $\mathfrak{F}$ in Z reduzibel, so müsste sich nach dem Vorhergehenden eine mit $\mathfrak{F}$ vertauschbare Matrix L angeben lassen, die mindestens zwei verschiedene charak-

* Vergl. A., p. 172.

† Vergl. L. E. DICKSON, loc. cit.

teristische Wurzeln w und w' besitzt. Versteht man dann unter $w_1, w_2, \cdots, w_{r-1}$ irgend welche $r-1$ Grössen, die unter einander und auch von w und w' verschieden sind, und setzt man

$$L_1 = w_1 E_f, \; L_2 = w_2 E_f, \; \cdots, \; L_{r-1} = w_{r-1} E_f,$$

so wird die Matrix

$$P_1 = \begin{Vmatrix} L & 0 & \cdots & 0 \\ 0 & L_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & L_{r-1} \end{Vmatrix}$$

mit der Gruppe $\mathfrak{F}'$ vertauschbar. Eine zu P_1 ähnliche Matrix P wäre dann mit der Gruppe $\mathfrak{G}$ vertauschbar. Diese Matrix würde jedoch ebenso wie P_1 mindestens $r+1$ verschiedene charakteristische Wurzeln $w, w', w_1, \cdots, w_{r-1}$ besitzen. Dies widerspricht aber der Annahme, die über die Zahl r gemacht worden ist. — Ebenso zeigt man, dass auch die Gruppen $\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_{r-1}$ im Bereiche Z irreduzibel sind.

Wir erhalten den Satz:

IV. *Jede in einem Körper Ω irreduzible Gruppe $\mathfrak{G}$ linearer Substitutionen ist im Bereich aller Zahlen vollständig reduzibel. Hat die allgemeinste mit $\mathfrak{G}$ vertauschbare Matrix genau r verschiedene charakteristische Wurzeln, so zerfällt $\mathfrak{G}$ im Bereich aller Zahlen in r irreduzible Gruppen desselben Grades. Diese r Gruppen lassen sich auch in r konjugierten algebraischen Körpern über Ω rational darstellen.*

Ferner gilt der Satz:

V. *Zwei in einem Körper Ω irreduzible Gruppen $\mathfrak{G}$ und $\mathfrak{G}_1$, die entweder isomorph oder einer dritten Gruppe homomorph sind, sind dann und nur dann äquivalent, wenn sie im Bereiche Z aller Zahlen einen irreduziblen Bestandteil gemeinsam haben.*

Denn enthalten $\mathfrak{G}$ und $\mathfrak{G}_1$ in Z einen irreduziblen Bestandteil gemeinsam, so sind sie als vollständig reduzible Gruppen unter einander verkettet und folglich nach Satz I* auch äquivalent.

§ 4.

Die zu der Gruppe $\mathfrak{G}$ gehörenden im Bereiche Z irreduziblen Gruppen $\mathfrak{F}, \mathfrak{F}_1, \cdots, \mathfrak{F}_{r-1}$ brauchen keineswegs in dem Sinne von einander verschieden zu sein, dass nicht zwei von ihnen einander äquivalent sind. Es gilt aber der Satz:

VI. *Es möge die im Körper Ω irreduzible Gruppe $\mathfrak{G}$ im Bereiche Z aller Zahlen in die irreduziblen Gruppen $\mathfrak{F}, \mathfrak{F}_1, \cdots, \mathfrak{F}_{r-1}$ zerfallen. Sind dann unter diesen Gruppen genau l vorhanden, von denen nicht zwei einander äquivalent sind, so ist $r/l = m$ eine ganze Zahl und es sind unter den Gruppen $\mathfrak{F}, \mathfrak{F}_1, \cdots, \mathfrak{F}_{r-1}$ je m einander äquivalent.*

Dies ergiebt sich fast unmittelbar aus dem Satz, der besagt, dass zwei in Z irreduzible einander isomorphe Gruppen dann und nur dann einander äquivalent sind, wenn je zwei einander entsprechende Substitutionen der beiden Gruppen dieselbe Spur besitzen.* — Es mögen nämlich die Spuren der Substitutionen der Gruppe $\mathfrak{F}$ zusammen mit den Zahlen von Ω einen Körper Ω' erzeugen. Da nun $\mathfrak{F}$ im Körper $\Omega(\rho)$ rational darstellbar ist, so ist Ω' jedenfalls ein Teilkörper von $\Omega(\rho)$ und daher bekanntlich selbst ein algebraischer Körper über Ω. Es möge etwa Ω' aus Ω durch Adjunktion der Grösse $\chi = \chi(\rho)$ von $\Omega(\rho)$ hervorgehen, so dass also $\Omega' = \Omega(\chi)$ wird. Dann erzeugen die Spuren der Substitutionen der Gruppe $\mathfrak{F}_\lambda$, die aus $\mathfrak{F}$ durch die Permutation $\rho | \rho_\lambda$ hervorgeht, zusammen mit den Zahlen von Ω den durch die Zahl $\chi_\lambda = \chi(\rho_\lambda)$ bestimmten Körper $\Omega(\chi_\lambda)$. Auf Grund des erwähnten Satzes über die Äquivalenz zweier irreduzibler Gruppen schliesst man sofort, dass $\mathfrak{F}_\varkappa$ und $\mathfrak{F}_\lambda$ dann und nur dann äquivalente Gruppen sind, wenn $\chi_\varkappa = \chi_\lambda$ wird. Genügt aber χ im Körper Ω einer irreduziblen Gleichung des Grades l, so ist bekanntlich l ein Divisor des Grades r des algebraischen Körpers $\Omega(\rho)$ über Ω und es sind unter den Zahlen $\chi, \chi_1, \cdots, \chi_{r-1}$ genau l von einander verschieden und je $m = r/l$ einander gleich. Hieraus folgt aber unser Satz.

Man beweist ferner ganz ähnlich wie bei der analogen Betrachtung A., p. 173:

VII. *Lässt sich die Gruppe $\mathfrak{F}$ in einem algebraischen Körper $\Omega(\sigma)$ des Grades s über Ω rational darstellen, so muss s durch r teilbar sein.*

Der früher bestimmte Körper $\Omega(\rho)$ ist also ein algebraischer Körper *kleinsten Grades* über Ω, in dem die Gruppe $\mathfrak{F}$ rational darstellbar ist.

Ebenso wie für endliche Gruppen (vergl. A., p. 171) gilt auch allgemein der Satz:

VIII. *Die Zahl $m = r/l$ ist ein Divisor des Grades f der Gruppe $\mathfrak{F}$.*

Der Beweis lässt sich mit Hilfe einer allgemeinen Satzes führen, der auch an und für sich von Wichtigkeit ist:

IX. *Man habe eine Gruppe $\mathfrak{H}$ linearer Substitutionen des Grades h, die im Bereiche Z aller Zahlen vollständig reduzibel ist. Unter den irreduziblen Bestandteilen von $\mathfrak{H}$ mögen genau l vorhanden sein, von denen nicht zwei einander äquivalent sind, etwa die Gruppen $\mathfrak{F}_0, \mathfrak{F}_1, \cdots, \mathfrak{F}_{l-1}$; hierbei soll keine dieser Gruppen aus lauter Nullen bestehen. Ist dann f_λ der Grad der Gruppe $\mathfrak{F}_\lambda$, so sind unter den Substitutionen von $\mathfrak{H}$ genau*

$$n = f_0^2 + f_1^2 + \cdots + f_{l-1}^2$$

linear unabhängig. Genügen etwa die Substitutionen $H_1, H_2, \cdots, H_n$ von $\mathfrak{H}$ dieser Bedingung, so sei $H = z_1 H_1 + z_2 H_2 + \cdots + z_n H_n$ die allgemeine Substitution von $\mathfrak{H}$ und speziell sei

*G. FROBENIUS und I. SCHUR, loc. cit., Satz II.

$$H_\alpha H_\beta = \sum_{\gamma=1}^{n} a_{\gamma\alpha\beta} H_\gamma. \tag{7}$$

Bezeichnet man dann mit S_α die Matrix

$$S_\alpha = (a_{\kappa\alpha\lambda}) \qquad (\kappa, \lambda = 1, 2, \cdots, n),$$

so bilden die Matrizen $S = z_1 S_1 + z_2 S_2 + \cdots + z_n S_n$ eine der Gruppe $\mathfrak{H}$ isomorphe Gruppe $\mathfrak{S}$, die ebenfalls vollständig reduzibel ist. Unter den irreduziblen Bestandteilen von $\mathfrak{S}$ kommen wieder nur die l Gruppen $\mathfrak{F}_0, \mathfrak{F}_1, \cdots, \mathfrak{F}_{l-1}$ vor, und zwar enthält $\mathfrak{S}$ die Gruppe $\mathfrak{F}_\lambda$ genau f_λ Mal.

Auf die Tatsache, dass die Gruppe $\mathfrak{S}$ zugleich mit der Gruppe $\mathfrak{H}$ vollständig reduzibel ist, hat bereits Herr TABER in seiner in der Einleitung zitierten Arbeit aufmerksam gemacht. Was hier also hinzugefügt wird, ist nur die Bestimmung der irreduziblen Bestandteile von $\mathfrak{S}$. Ich bemerke noch, dass unser Satz für den besonderen Fall, dass die Gruppe $\mathfrak{H}$ irreduzibel ist, bereits von Herrn BURNSIDE* bewiesen worden ist.

Der Beweis des Satzes IX ergiebt sich folgendermassen.

Es möge der Substitution H_α von $\mathfrak{H}$ in $\mathfrak{F}_\lambda$ die Substitution $F_\alpha^{(\lambda)}$ entsprechen. Sind dann $x_1, x_2, \cdots, x_n$ unabhängige Variable und setzt man

$$X^{(\lambda)} = x_1 F_1^{(\lambda)} + x_2 F_2^{(\lambda)} + \cdots + x_n F_n^{(\lambda)},$$

so sind, weil unter den irreduziblen Gruppen $\mathfrak{F}_0, \mathfrak{F}_1, \cdots, \mathfrak{F}_{l-1}$ nicht zwei einander äquivalent sind, die $f_0^2 + f_1^2 + \cdots + f_{l-1}^2$ Koeffizienten der l Matrizen $X^{(0)}, X^{(1)}, \cdots, X^{(l-1)}$, als Funktionen der x_ν betrachtet, linear unabhängig.† Andererseits sind unter den h^2 Koeffizienten der Matrix

$$x_1 H_1 + x_2 H_2 + \cdots + x_n H_n$$

genau n linear unabhängig. Hieraus schliesst man wegen der über $\mathfrak{H}$ gemachten Annahmen sofort, dass $n = f_0^2 + f_1^2 + \cdots + f_{l-1}^2$ sein muss. Man erkennt auch leicht, dass das durch die Gleichungen (7) definierte System hyperkomplexer Grössen nach der von Herrn FROBENIUS‡ eingeführten Bezeichnungsweise ein DEDEKIND'scher System ist, d. h. es muss die Determinante D der n^2 Grössen

$$p_{\alpha\beta} = \sum_{\kappa,\lambda} a_{\kappa\lambda\kappa} a_{\lambda\alpha\beta} \qquad (\alpha, \beta = 1, 2, \cdots, n)$$

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* *On the arithmetical nature of the coefficients in a group of linear substitutions of finite order* (second paper), **Proceedings of the London Mathematical Society, ser. 2, vol. 4 (1906), p. 1.**

† Vergl. G. FROBENIUS und I. SCHUR, loc. cit., Satz I.

‡ *Theorie der hyperkomplexen Grössen*, **Sitzungsberichte der Berliner Akademie, 1903, p. 504 und p. 634.**

von Null verschieden sein.* Daher ist die durch die Matrizen

$$x_1 S_1 + x_2 S_2 + \cdots + x_n S_n$$

gebildete Gruppe $\overline{\mathfrak{S}}$ eine vollständig reduzible Gruppe, die jeden ihrer irreduziblen Bestandteile genau so oft enthält wie sein Grad angiebt.† Dasselbe gilt natürlich auch für die mit $\mathfrak{H}$ isomorphe Gruppe $\mathfrak{S}$. Nun ist aber auch

$$F_\alpha^{(\lambda)} F_\beta^{(\lambda)} = \sum_\gamma a_{\gamma\alpha\beta} F_\gamma^{(\lambda)}. \tag{8}$$

Setzt man

$$F_\alpha^{(\lambda)} = (k_{\mu\nu}^{(\alpha)}) \qquad (\mu, \nu = 1, 2, \cdots, f_\lambda)$$

und bezeichnet mit P_ν die Matrix

$$P_\nu = \begin{bmatrix} k_{1\nu}^{(1)} & k_{1\nu}^{(2)} & \cdots & k_{1\nu}^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ k_{f_\lambda \nu}^{(1)} & k_{f_\lambda \nu}^{(2)} & \cdots & k_{f_\lambda \nu}^{(n)} \end{bmatrix},$$

so lassen sich die Gleichungen (8) auch in der Form

$$F_\alpha^{(\lambda)} P_\nu = P_\nu S_\alpha$$

schreiben. Da nun die P_ν nicht sämtlich 0 sein können, so besagen diese Gleichungen (vergl. § 1), dass die Gruppe $\mathfrak{S}$ mit der Gruppe $\mathfrak{F}_\lambda$ verkettet ist. Nach Satz I muss daher $\mathfrak{F}_\lambda$ ein irreduzibler Bestandteil von $\mathfrak{S}$ sein und ist nach dem Gesagten genau f_λ Mal in $\mathfrak{S}$ enthalten.‡ Da dies für jedes λ gilt, so erhält man $f_0 + f_1 + \cdots + f_{l-1}$ irreduzible Bestandteile von $\mathfrak{S}$, die aneinander gereiht eine Gruppe des Grades $f_0^2 + f_1^2 + \cdots + f_{l-1}^2 = n$ ergeben. Da aber n der Grad von $\mathfrak{S}$ ist, so kann $\mathfrak{S}$ keinen weiteren irreduziblen Bestandteil enthalten. §

Ist nun speziell $\mathfrak{H} = \mathfrak{G}$, wie früher, eine im Körper Ω irreduzible Gruppe, so wird $f_0 = f_1 = \cdots = f_{l-1} = f$, also $n = lf^2$. Ferner gehören die Zahlen $a_{\gamma\alpha\beta}$ in diesem Fall ebenfalls den Körper Ω an. Denkt man sich nun die in Ω ratio-

* Vergl. TABER, loc. cit. — Herr TABER hat noch die wichtige Bemerkung hinzugefügt, dass für jede beliebige Gruppe $\mathfrak{H}$ die Determinante D dann und nur dann verschwindet, wenn die mit Hilfe der Spuren $q_{\alpha\beta}$ der Matrizen $H_\alpha H_\beta$ gebildete Determinante $\Delta = |q_{\alpha\beta}|$ gleich Null ist. Es lässt sich auch, wie bei dieser Gelegenheit hervorgehoben sei, zeigen, dass *die Determinanten D und Δ stets denselben Rang besitzen.*

† Dass die zu einem System hyperkomplexer Grössen mit nicht verschwindender Determinante D gehörende Gruppe $\mathfrak{S}$ diese Eigenschaft besitzt, hat zuerst Herr MOLIEN in seiner Arbeit *Über Systeme höherer complexer Zahlen*, Mathematische Annalen, Bd. 41 (1893), p. 83, gezeigt. Vergl. auch CARTAN, *Sur les groupes bilinéaires et les systèmes de nombres complexes*, Annales de Toulouse, t. XII (1898), p. 1 und FROBENIUS, loc. cit.

‡ Vergl. auch FROBENIUS, loc. cit., p. 536.

§ Erzeugen die Spuren der Substitutionen der (vollständig reduziblen) Gruppe $\mathfrak{H}$ den Rationalitätsbereich Ω', so gehören auch die Zahlen $a_{\gamma\alpha\beta}$ dem Körper Ω' an (vergl. BURNSIDE, loc. cit., p. 4). Auf Grund dieser Bemerkung kann man weiter schliessen, dass die Gruppe $\mathfrak{H}$ in einem algebraischen Körper über Ω' rational darstellbar ist.

nale Gruppe $\mathfrak{S}$ inbezug auf Ω in irreduzible Bestandteile $\mathfrak{G}_1, \mathfrak{G}_2, \cdots, \mathfrak{G}_p$ zerlegt, so muss jede dieser Gruppen der Gruppe $\mathfrak{G}$ äquivalent sein. Denn wäre dies für die Gruppe $\mathfrak{G}_\lambda$ nicht der Fall, so wären nach Satz V die irreduziblen Bestandteile von $\mathfrak{G}_\lambda$ inbezug auf den Körper Z aller Zahlen von denjenigen der Gruppe $\mathfrak{G}$ wesentlich verschieden, folglich müsste auch die Gruppe $\mathfrak{S}$, im Körper Z zerlegt, einen irreduziblen Bestandteil enhalten, der keiner der Gruppen $\mathfrak{F}, \mathfrak{F}_1, \cdots, \mathfrak{F}_{r-1}$ äquivalent ist. Dies widerspricht aber dem soeben Bewiesenen. Daher ist

$$n = lf^2 = pg = pmlf,$$

und hieraus folgt, dass die Zahl m in der Tat ein Divisor der Zahl f ist.

§ 5.

Es sei nun $\mathfrak{K}$ eine beliebige im Körper Ω rationale Gruppe linearer Substitutionen des Grades k. Diese Gruppe denken wir uns zunächst im Körper Ω in irreduzible Bestandteile zerlegt, es sei etwa

$$\begin{Vmatrix} \mathfrak{G}_{11} & 0 & \cdots & 0 \\ \mathfrak{G}_{21} & \mathfrak{G}_{22} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \mathfrak{G}_{p1} & \mathfrak{G}_{p2} & \cdots & \mathfrak{G}_{pp} \end{Vmatrix}$$

ein in Ω rationale, mit $\mathfrak{K}$ äquivalente Gruppe, wo $\mathfrak{G}_{11}, \mathfrak{G}_{22}, \cdots, \mathfrak{G}_{pp}$ in Ω irreduzibel sind. Es möge ferner die Gruppe $\mathfrak{G}_{\alpha\alpha}$, im Bereiche Z aller Zahlen zerlegt, die irreduziblen Bestandteile $\mathfrak{F}_{\alpha 1}, \mathfrak{F}_{\alpha 2}, \cdots, \mathfrak{F}_{\alpha r_\alpha}$ aufweisen. Dann repräsentieren die sich so ergebenden $r_1 + r_2 + \cdots + r_p$ Gruppen $\mathfrak{F}_{\alpha\beta}$ in ihrer Gesamtheit die irreduziblen Bestandteile der Gruppe $\mathfrak{K}$. Auf Grund der früher gewonnenen Resultate kann man nun den Satz aussprechen:

X. *Man habe eine Gruppe $\mathfrak{K}$ linearer homogener Substitutionen, die in einem Zahlkörper Ω rational ist. Ist dann $\mathfrak{F}$ einer der irreduziblen Bestandteile, in die $\mathfrak{K}$ im Bereiche aller Zahlen zerfällt, so erzeugen die Spuren der Substitutionen von $\mathfrak{F}$ zusammen mit den Zahlen von Ω einen algebraischen Körper über Ω, ferner lässt sich $\mathfrak{F}$ in einem algebraischen Körper $\Omega(\rho)$ über Ω rational darstellen. Wählt man den Grad r des Körpers $\Omega(\rho)$ möglichst klein, so seien $\rho_1, \rho_2, \cdots, \rho_{r-1}$ die zu ρ konjugierten Zahlen und es möge $\mathfrak{F}$ in $\mathfrak{F}_\lambda$ übergehen, wenn in allen Substitutionskoeffizienten von $\mathfrak{F}$ die Zahl ρ durch ρ_λ ersetzt wird. Dann ist die Gruppe*

$$\mathfrak{F}' = \begin{Vmatrix} \mathfrak{F} & 0 & \cdots & 0 \\ 0 & \mathfrak{F}_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \mathfrak{F}_{r-1} \end{Vmatrix}$$

einer in Ω rationalen und irreduziblen Gruppe $\mathfrak{G}$ äquivalent, die in $\mathfrak{K}$ als irreduzibler Bestandteil inbezug auf Ω enthalten ist.

Ich stelle nun folgende Betrachtung an. Es sei

$$(K) \qquad \bar{x}_\kappa = \sum_{\lambda=1}^{k} c_{\kappa\lambda} x_\lambda$$

eine beliebige Substitution von $\mathfrak{K}$. Ist $\phi(x_1, x_2, \cdots, x_n)$ irgend eine homogene Form der Variabeln $x_1, x_2, \cdots, x_n$, so möge

$$\phi(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) = \bar{\phi}(x_1, x_2, \cdots, x_n)$$

gesetzt werden. Man denke sich nun eine der Gruppe $\mathfrak{K}$ homomorphe Gruppe $\mathfrak{A}$ des Grades a gegeben, in der der Substitution K von $\mathfrak{K}$ die Substitution $A = (a_{\alpha\beta})$ entsprechen möge. Es kann dann eintreten, dass sich gewisse a, nicht notwendig linear unabhängige, Formen $\phi_1, \phi_2, \cdots, \phi_a$ gleicher Ordnung angeben lassen, so dass

$$(9) \qquad \bar{\phi}_\alpha = \sum_{\beta=1}^{a} a_{\alpha\beta} \phi_\beta$$

wird. Ist dies für jede Substitution K von $\mathfrak{K}$ der Fall, so will ich sagen: *$\phi_1, \phi_2, \cdots, \phi_a$ bilden ein zur Gruppe $\mathfrak{A}$ gehörendes Invariantensystem von $\mathfrak{K}$.* Sind insbesondere die Formen $\phi_1, \phi_2, \cdots, \phi_a$ unter einander linear unabhängig, und weiss man, dass für jede Substitution K von $\mathfrak{K}$ Gleichungen der Form (9) bestehen, so sind die Koeffizienten $a_{\alpha\beta}$ eindeutig bestimmt. Die Substitutionen $A = (a_{\alpha\beta})$ bilden dann von selbst eine der Gruppe $\mathfrak{K}$ homomorphe Gruppe. In diesem Falle sage ich auch: *$\phi_1, \phi_2, \cdots, \phi_a$ bilden ein Invariantensystem von $\mathfrak{K}$ mit der Transformationsgruppe $\mathfrak{A}$.* Für die Gleichungen (9) schreibe ich in jedem Falle auch kürzer

$$(\bar{\phi}_\alpha) = \mathfrak{A}(\phi_\alpha).$$

Es gilt nun der für viele Anwendungen nützliche Satz:

XI. *Es sei $\mathfrak{A}$ eine im Körper Ω irreduzible Gruppe, die der in Ω rationalen Gruppe $\mathfrak{K}$ homomorph ist. Hat man dann ein zur Gruppe $\mathfrak{A}$ gehörendes Invariantensystem $\phi_1, \phi_2, \cdots, \phi_a$ von $\mathfrak{K}$ mit Koeffizienten aus dem Körper Ω, so sind die Formen $\phi_1, \phi_2, \cdots, \phi_a$ entweder alle Null oder sie sind unter einander linear unabhängig.*

Dies ist leicht zu beweisen. Denn es seien die Formen ϕ_α nicht alle Null; man bezeichne die Anzahl der linear unabhängigen unter ihnen mit r. Dann lassen sich also r linear unabhängige Formen $\psi_1, \psi_2, \cdots, \psi_r$ mit Koeffizienten aus dem Körper Ω angeben, so dass Gleichungen der Form

$$(10) \qquad \psi_\rho = \sum_{\alpha=1}^{a} p_{\rho\alpha} \phi_\alpha, \qquad \phi_\alpha = \sum_{\rho=1}^{r} q_{\alpha\rho} \psi_\rho$$

bestehen, wo die $p_{\rho\alpha}$ und $q_{\alpha\rho}$ Zahlen von Ω sind. Aus (10) folgt, wenn

$$r_{\rho\sigma} = \sum_{\alpha,\beta=1}^{a} p_{\rho\alpha} a_{\alpha\beta} q_{\beta\sigma}$$

gesetzt wird,

$$\overline{\psi}_\rho = \sum_{\sigma=1}^{r} r_{\rho\sigma} \psi_\sigma .$$

Bezeichnet man daher die durch die Substitutionen $R = (r_{\rho\sigma})$ gebildete Gruppe mit $\mathfrak{R}$, so erscheinen die Formen $\psi_1, \psi_2, \cdots, \psi_r$ als ein Invariantensystem von $\mathfrak{K}$ mit der Transformationsgruppe $\mathfrak{R}$. Andererseits wird aber

$$\overline{\phi}_\alpha = \sum_\rho q_{\alpha\rho} \overline{\psi}_\rho = \sum_{\rho,\sigma} q_{\alpha\rho} r_{\rho\sigma} \psi_\sigma = \sum_\beta a_{\alpha\beta} \phi_\beta = \sum_{\beta,\sigma} a_{\alpha\beta} q_{\beta\sigma} \psi_\sigma ,$$

also ist

$$\sum_{\rho=1}^{r} q_{\alpha\rho} r_{\rho\sigma} = \sum_{\beta=1}^{a} a_{\alpha\beta} q_{\beta\sigma} .$$

Setzt man die Matrix $(q_{\alpha\rho})$ gleich Q, so wird auf Grund der in § 1 festgesetzten Bezeichnungsweise

$$Q\mathfrak{R} = \mathfrak{A}Q .$$

Da nun $\mathfrak{A}$ in Ω irreduzibel ist, so muss nach Satz I der Rang von Q gleich a sein. Dies erfordert aber, dass $r = a$ wird.*

Es sei nun $\mathfrak{F}$ wie früher ein irreduziblen Bestandteil des Grades f von $\mathfrak{K}$ inbezug auf den Bereich aller Zahlen. Ich nehme insbesondere an, dass sich f (nicht identisch verschwindende) lineare homogene Formen $u_1, u_2, \cdots, u_f$ der x_κ angeben lassen, die ein zu $\mathfrak{F}$ gehörendes Invariantensystem von $\mathfrak{K}$ bilden; es sei auch noch verlangt, dass $u_1, \cdots, u_f$ von gewissen gegebenen a linearen homogenen Formen $y_1, y_2, \cdots, y_a$ mit in Ω rationalen Koeffizienten linear unabhängig sein sollen. Hat nun $\mathfrak{F}$ diese Eigenschaft, so gilt dies auch für jede mit $\mathfrak{F}$ äquivalente Gruppe. Denkt man sich insbesondere $\mathfrak{F}$ durch eine äquivalente Gruppe ersetzt, die in dem oben genannten Körper $\Omega(\rho)$ rational ist, so lassen sich die Linearformen $u_1, u_2, \cdots, u_f$ auch so wählen, dass ihre Koeffizienten ebenfalls Zahlen von $\Omega(\rho)$ werden, ohne dass die Linearformen aufhören den früher gestellten Forderungen zu genügen. Geht dann $u_\alpha = u_{\alpha 0}$ in $u_{\alpha\nu}$ über, wenn ρ durch ρ_ν ersetzt wird, so bilden die f Linearformen $u_{1\nu}$, $u_{2\nu}, \cdots, u_{f\nu}$ ein zur Gruppe $\mathfrak{F}_\nu$, ferner die $rf = g$ Linearformen $u_{\alpha\nu}$ zusammengenommen ein zur Gruppe $\mathfrak{F}'$ gehörendes Invariantensystem von $\mathfrak{K}$.

Setzt man nun weiter

$$z_{\alpha\mu} = \sum_{\nu=0}^{r-1} \rho_\nu^\mu u_{\alpha\nu} \quad (\alpha = 1, 2, \cdots, f;\ \mu = 0, 1, \cdots, r-1), \tag{11}$$

* Nimmt man $\mathfrak{A}$ nicht als irreduzibel an, so ergiebt sich, dass $\mathfrak{A}$ einer in Ω rationalen Gruppe der Form

$$\begin{pmatrix} \mathfrak{R} & \mathfrak{Q} \\ 0 & \mathfrak{P} \end{pmatrix}$$

äquivalent ist.

so werden die Koeffizienten dieser $rf = g$ Linearformen der $x_\varkappa$, als symmetrische Funktionen der ρ_ν, Zahlen des Körpers Ω.* Ist ferner T die Koeffizientenmatrix der Formen $z_{a\mu}$, als Funktionen der $u_{a\nu}$ aufgefasst, und setzt man

$$T\mathfrak{F}'T^{-1} = \mathfrak{G},$$

so erkennt man sofort, dass die $z_{a\mu}$ ein zur Gruppe $\mathfrak{G}$ gehörendes Invariantensystem von $\mathfrak{K}$ bilden. Die Gruppe $\mathfrak{G}$ ist aber (vergl. § 1) eine in Ω rationale und irreduzible Gruppe. Folglich müssen nach Satz XI die Linearformen $z_{a\mu}$, da sie offenbar nicht alle Null sein können, *unter einander linear unabhängig sein.*

§ 6.

Ich wende mich nun zur Betrachtung des in der Einleitung erwähnten Satzes von Herrn A. LOEWY.

Der Gedankengang des Herrn LOEWY lässt sich folgendermassen formulieren. Man habe eine im Körper Ω rationale und in diesem Körper vollständig reduzible Gruppe $\mathfrak{A}$ des Grades a, die zu unserer Gruppe $\mathfrak{K}$ homomorph ist. Es mögen sich ferner a unabhängige Linearformen $y_1, y_2, \cdots, y_a$ der $x_\varkappa$ mit Koeffizienten aus dem Körper Ω bestimmen lassen, die ein Invariantensystem von $\mathfrak{K}$ mit der Transformationsgruppe $\mathfrak{A}$ bilden. Wir verlangen weiter, dass die Gruppe $\mathfrak{A}$ noch folgender Bedingung genügen soll: es soll sich keine in Ω irreduzible mit $\mathfrak{K}$ homomorphe Gruppe $\mathfrak{G}$ des grades g angeben lassen, zu der ein lineares Invariantensystem $z_1, z_2, \cdots, z_g$ von $\mathfrak{K}$ mit Koeffizienten aus Ω in der Weise gehört, dass die $a + g$ Linearformen $y_1, y_2, \cdots, y_a, z_1, \cdots, z_g$ unter einander linear unabhängig werden. Dann beweist Herr LOEWY:

"Ist $\overline{\mathfrak{A}}$ eine zweite Gruppe, welche genau denselben Forderungen genügt, wie die Gruppe $\mathfrak{A}$, so sind $\mathfrak{A}$ und $\overline{\mathfrak{A}}$ äquivalente Gruppen. Bestimmt man ferner, was jedenfalls möglich ist, zwei zu $\mathfrak{K}$ äquivalente in Ω rationale Gruppen $\mathfrak{K}'$ und $\overline{\mathfrak{K}}$ der Form

$$\mathfrak{K}' = \begin{pmatrix} \mathfrak{A} & 0 \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}, \qquad \overline{\mathfrak{K}} = \begin{pmatrix} \overline{\mathfrak{A}} & 0 \\ \overline{\mathfrak{C}} & \overline{\mathfrak{D}} \end{pmatrix},$$

so sind auch $\mathfrak{D}$ und $\overline{\mathfrak{D}}$ äquivalente Gruppen."†

Man kann daher $\mathfrak{A}$ als die grösste zu $\mathfrak{K}$ gehörende vollständig reduzible Gruppe inbezug auf Ω bezeichnen.

Tritt nun an Stelle des Körpers Ω ein anderer Körper Ω', der ebenso wie Ω alle Koeffizienten der Substitutionen von $\mathfrak{K}$ umfasst, so existiert auch inbezug auf Ω' eine zu $\mathfrak{K}$ gehörende grösste vollständig reduzible Gruppe $\mathfrak{A}'$.

* Vergl. L. E. DICKSON, loc. cit.

† Herr LOEWY macht in seiner Arbeit noch die Voraussetzung, dass die Determinanten der Substitutionen von $\mathfrak{K}$ nicht verschwinden, und dass die reziproke Substitution jeder Substitution von $\mathfrak{K}$ wieder in $\mathfrak{K}$ enthalten ist. Das von Herrn STICKELBERGER (vergl. das in der Einleitung angeführte Zitat) mitgeteilte Beweisverfahren gestattet aber, den LOEWY'schen Satz auch für eine allgemeine Gruppe $\mathfrak{K}$ zu beweisen.

Ich will nun zeigen, dass $\mathfrak{A}$ und $\mathfrak{A}'$ äquivalente Gruppen sind.

Es genügt offenbar, dies allein für den Fall zu beweisen, dass Ω' den Bereich Z aller Zahlen bedeutet. Der Beweis gestaltet sich nun folgendermassen.

Da jede im Körper Ω irreduzible Gruppe im Körper Z vollständig reduzibel ist, so ist die in Ω vollständig reduzible Gruppe $\mathfrak{A}$ in Z ebenfalls vollständig reduzibel. Wir haben also nur zu zeigen, dass $\mathfrak{A}$ auch, im Körper Z betrachtet, die Eigenschaften einer grössten vollständig reduziblen Gruppe besitzt.

Wäre dies nun nicht der Fall, so müsste sich eine zu $\mathfrak{R}$ homomorphe in Z irreduzible Gruppe $\mathfrak{F}$ des Grades f angeben lassen, zu der ein lineares Invariantensystem $u_1, u_2, \cdots, u_f$ gehört, so dass die $a+f$ Linearformen $y_1, \cdots, y_a$, $u_1, \cdots, u_f$ unter einander linear unabhängig sind. Die Gruppe $\mathfrak{F}$ wird dann ein irreduzibler Bestandteil von $\mathfrak{R}$ inbezug auf Z. Man bestimme nun, wie im vorigen Paragraphe, die aus $\mathfrak{F}$ hervorgehende inbezug auf Ω irreduzible Gruppe $\mathfrak{G}$. Dann gehört auch zu dieser, wie wir gesehen haben, ein lineares Invariantensystem $z_{a\mu}$ mit in Ω rationalen Koeffizienten. Da aber $\mathfrak{A}$ inbezug auf Ω eine grösste vollständig reduzible Gruppe sein soll, so dürften die Linearformen $z_{a\mu}$ und $y_1, y_2, \cdots, y_a$ nicht unter einander linear unabhängig sein. Hieraus folgt aber (wegen der Irreduzibilität von $\mathfrak{G}$ inbezug auf Ω) nach einem von Herrn STICKELBERGER, loc. cit., bewiesenen Satze, dass jede der Funktionen $z_{a\mu}$ durch $y_1, y_2, \cdots, y_a$ linear darstellbar sein muss. Es ist aber, wie aus den Gleichungen (11) hervorgeht, jede der Funktionen $u_1, u_2, \cdots, u_f$ eine lineare homogene Form der $z_{a\mu}$ und wäre folglich auch durch $y_1, y_2, \cdots, y_a$ linear und homogen darstellbar. Dies führt aber auf einen Widerspruch.

Ist wie früher

$$\mathfrak{R}' = \begin{pmatrix} \mathfrak{A} & 0 \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}$$

eine zu $\mathfrak{R}$ äquivalente in Ω rationale Gruppe, so gehört auch zur Gruppe $\mathfrak{D}$ inbezug auf Ω eine grösste vollständig reduzible Gruppe u. s. f. So erhält Herr LOEWY die zu $\mathfrak{R}$ inbezug auf Ω gehörenden auf einander folgenden grössten vollständig reduziblen Gruppen $\mathfrak{A}, \mathfrak{A}_1, \cdots, \mathfrak{A}_\mu$. Aus unserer Betrachtung ergiebt sich aber, dass diese Gruppen in dem in der Einleitung genauer erläuterten Sinne von der Wahl des Körpers Ω unabhängig sind.

Man kann dieses Resultat auch folgendermassen aussprechen:

XII. *Ist $\mathfrak{R}$ eine in einem gegebenen Zahlkörper Ω rationale Gruppe linearer homogener Substitutionen und zerlegt man $\mathfrak{R}$ im Bereich aller Zahlen unter Hervorhebung der zu $\mathfrak{R}$ gehörenden auf einander folgenden grössten vollständig reduziblen Gruppen, so ist jede dieser Gruppen im Körper Ω rational darstellbar.*

BERLIN, *im März* 1908.

PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVED SURFACES*

(FOURTH MEMOIR)

BY

E. J. WILCZYNSKI

The theory of surfaces, as developed by the author in the preceding three memoirs, was based upon the assumption that the surface considered was referred to its asymptotic curves. The complete system of invariants and covariants was set up on that hypothesis. In order to make the theory directly applicable to a surface whose asymptotic lines are not known, as well as for the sake of formal generality, it becomes necessary to develop the theory of invariants and covariants of a system of two linear homogeneous partial differential equations of the second order with one dependent and two independent variables in its most general form. It is the purpose of the present paper to develop the essentials of this theory. The calculations, some of which are very long, will only be indicated. The final expressions for the invariants, however, are as simple and elegant as can be desired.

§ 1. *Reduction of the given system of partial differential equations to its normal form.*

Consider the system of partial differential equations

$$\begin{aligned} \Omega &\equiv Ay_{uu} + 2By_{uv} + Cy_{vv} + Ly_u + My_v + Ny = 0, \\ \Omega' &\equiv A'y_{uu} + 2B'y_{uv} + C'y_{vv} + L'y_u + M'y_v + N'y = 0, \end{aligned} \tag{1}$$

where $A, \cdots, N'$ are functions of u and v, and where the subscripts indicate differentiations;

$$y_u = \frac{\partial y}{\partial u}, \qquad y_v = \frac{\partial y}{\partial v}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \quad \text{etc.} \quad .$$

If λ, μ, ν, ρ denote four functions of u and v whose determinant $\lambda\rho - \mu\nu$ is different from zero, the system

$$\lambda\Omega + \mu\Omega' = 0, \qquad \nu\Omega + \rho\Omega' = 0 \tag{2}$$

* Presented to the Society (Chicago), April 18, 1908.

is equivalent to (1). Put

$$(3)\quad AC - B^2 = I, \qquad A'C' - B'^2 = I', \qquad AC' + A'C - 2BB' = J,$$

and denote by $\bar{I}$, $\bar{I}'$, $\bar{J}$ the corresponding quantities for system (2). Then

$$(4)\quad \bar{I} = \lambda^2 I + \lambda\mu J + \mu^2 I', \qquad \bar{I}' = \nu^2 I + \nu\rho J + \rho^2 I'.$$

Consequently, if

$$K = J^2 - 4II' \neq 0,$$

λ, μ, ν, ρ may be chosen in such a way that their determinant does not vanish and so that

$$\bar{I} = \bar{I}' = 0,$$

while $\bar{J}$ will be different from zero. In fact, if $K \neq 0$,

$$\bar{K} = (\lambda\rho - \mu\nu)K$$

will also be different from zero, so that the same will be true of $\bar{J}$. It suffices for our purpose to choose $\lambda : \mu$ and $\nu : \rho$ as the two distinct roots of the quadratic

$$\alpha^2 I + \alpha\beta J + \beta^2 I' = 0.$$

By merely solving a quadratic equation, we may therefore substitute for a system of form (1) an equivalent system for which $I = I' = 0$, while J does not vanish, provided only that $J^2 - 4II'$ is not equal to zero for the original system. We shall say that the system (1) has been reduced to its *normal form*. This reduced form of system (1) is not unique; but, the most general reduced system may be obtained from any particular one by multiplying each of the equations of the system by an arbitrary function of u and v, and by interchanging the two equations.

Assume that system (1) is in its normal form. It may be verified directly, or it may be deduced from formulæ which will be developed later, that the normal form will not be disturbed by any transformation of the form

$$\bar{y} = \lambda(u, v)y, \qquad \bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v),$$

where λ, ϕ and ψ are arbitrary functions of u and v. We shall, therefore, always assume that the system (1) is given in its normal form, so that

$$AC - B^2 = 0, \qquad A'C' - B'^2 = 0, \qquad AC' + A'C - 2BB' \neq 0.$$

We may therefore put

$$(5)\quad \begin{aligned} A &= h^2, & B &= hk, & C &= k^2, \\ A' &= h'^2, & B' &= h'k', & C' &= k'^2, \\ \Delta &= hk' - h'k \neq 0, \end{aligned}$$

so that (1) assumes the form

$$\begin{aligned} h^2 y_{uu} + 2hky_{uv} + k^2 y_{vv} &= ry + sy_u + ty_v, \\ h'^2 y_{uu} + 2h'k' y_{uv} + k'^2 y_{vv} &= r'y + s'y_u + t'y_v. \end{aligned} \tag{6}$$

The case of a system of form (1), for which

$$J^2 - 4II' = 0,$$

may be excluded from consideration for the purposes of the projective theory of surfaces. For it easily follows from formulæ given later that the integral surfaces of such a system degenerate either into curves or developables. In the latter case moreover the system is involutory; so that the most general developable integral surface of such a system is not a projective transformation of any particular one. For the projective theory of surfaces then, such systems are without interest.

§2. *The integrability conditions.*

Equation (6) may be solved for two of the second derivatives, thus expressing them homogeneously and linearly in terms of y, y_u, y_v and the remaining derivative of the second order. Thus (6) may be, in general, solved for y_{uu} and y_{vv} in terms of y, y_u, y_v, y_{uv}. But this is not always the case, in fact it cannot be done if

$$hk' + h'k = 0.$$

In that case (6) might be solved for y_{uu} and y_{uv}. In order, however, to avoid the discussion of such special cases, and so as to preserve symmetry, we shall introduce the following symmetrical combination of the second derivatives:

$$w = hh'y_{uu} + (hk' + h'k)y_{uv} + kk'y_{vv}. \tag{7}$$

It is *always* possible to express y_{uu}, y_{uv}, y_{vv} as linear homogeneous functions of y, y_u, y_v and w. Put

$$\frac{h}{\Delta} = p, \qquad \frac{k}{\Delta} = q, \qquad \frac{h'}{\Delta} = p', \qquad \frac{k'}{\Delta} = q', \tag{8}$$

and

$$\begin{aligned} &a = rq'^2 + r'q^2, \quad b = sq'^2 + s'q^2, \quad c = tq'^2 + t'q^2, \quad d = -2qq', \\ &a' = -(rp'q' + r'pq), \quad b' = -(sp'q' + s'pq), \quad c' = -(tp'q' + t'pq), \quad d' = pq' + p'q, \\ &a'' = rp'^2 + r'p^2, \quad b'' = sp'^2 + s'p^2, \quad c'' = tp'^2 + t'p^2, \quad d'' = -2pp', \\ &d'^2 - dd'' = \frac{1}{\Delta^2}. \end{aligned} \tag{9}$$

Then

$$\begin{aligned} y_{uu} &= ay + by_u + cy_v + dw, \\ y_{uv} &= a'y + b'y_u + c'y_v + d'w, \\ y_{vv} &= a''y + b''y_u + c''y_v + d''w. \end{aligned} \tag{10}$$

There can be no relation of the form

$$\lambda y + \mu y_u + \nu y_v + \rho w = 0, \tag{11}$$

with non-vanishing coefficients in any case which is of interest in the projective theory of surfaces. In fact, the existence of a relation of form (11), together with equations (10) and those deducible therefrom by differentiation, would imply that the most general solution of (6) could contain no more than three arbitrary constants. This is excluded because in our geometrical interpretation such a system cannot be said to have an integral surface. *The non-existence of a relation of the form* (11) *may be expressed geometrically by saying that the four points* P_y, P_{y_u}, P_{y_v}, P_w *form a non-degenerate tetrahedron.*

We proceed to express w_u and w_v in terms of y, y_u, y_v and w. From (10) we find the two expressions for y_{uuv};

$$\begin{aligned} y_{uuv} &= (a_v + a'b + a''c)y + (b_v + bb' + b''c)y_u \\ &\quad + (c_v + a + bc' + cc'')y_v + (d_v + bd' + cd'')w + dw_v, \\ y_{uuv} &= (a'_u + ab' + a'c')y + (b'_u + a' + bb' + b'c')y_u \\ &\quad + (c'_u + b'c + c'^2)y_v + (d'_u + b'd + c'd')w + d'w_u. \end{aligned} \tag{12}$$

Similarly

$$\begin{aligned} y_{uvv} &= (a'_v + a'b' + a''c')y + (b'_v + b'^2 + b''c')y_u \\ &\quad + (c'_v + a' + b'c' + c'c'')y_v + (d'_v + b'd' + c'd'')w + d'w_v, \\ y_{uvv} &= (a''_u + ab'' + a'c'')y + (b''_u + a'' + bb'' + b'c'')y_u \\ &\quad + (c''_u + b''c + c'c'')y_v + (d''_u + b''d + c''d')w + d''w_u. \end{aligned} \tag{13}$$

From these equations we deduce

$$\begin{aligned} d'w_u - dw_v &= (a_v - a'_u + a'b - ab' + a''c - a'c')y + (b_v - b'_u - a' + b''c - b'c')y_u \\ &\quad + (c_v - c'_u + a + bc' - b'c + cc'' - c'^2)y_v \\ &\quad + (d_v - d'_u + bd' - b'd + cd'' - c'd')w, \\ d''w_u - d'w_v &= (a'_v - a''_u + a'b' - ab'' + a''c' - a'c'')y \\ &\quad + (b'_v - b''_u - a'' + b'^2 - bb'' + b''c' - b'c'')y_u \\ &\quad + (c'_v - c''_u + a' + b'c' - b''c)y_v + (d'_v - d''_u + b'd' - b''d + c'd'' - c''d')w, \end{aligned} \tag{14}$$

whence

$$w_u = \alpha y + \beta y_u + \gamma y_v + \delta w, \qquad w_v = \alpha' y + \beta' y_u + \gamma' y_v + \delta' w, \tag{15}$$

where

$$\frac{1}{\Delta^2}\alpha = d'(a_v - a'_u + a'b - ab' + a''c - a'c') - d(a'_v - a''_u + a'b' - ab'' + a''c' - a'c''),$$

$$
\begin{aligned}
\frac{1}{\Delta^2}\beta &= d'(b_v - b_u' - a' + b''c - b'c') - d(b_v' - b_u'' - a'' + b'^2 - bb'' + b''c' - b'c''),\\
\frac{1}{\Delta^2}\gamma &= d'(c_v - c_u' + a + bc' - b'c + cc'' - c'^2) - d(c_v' - c_u'' + a' + b'c' - b''c),\\
\frac{1}{\Delta^2}\delta &= d'(d_v - d_u' + bd' - b'd + cd'' - c'd') - d(d_v' - d_u'' + b'd' - b''d + c'd'' - c''d'),\\
\frac{1}{\Delta^2}\alpha' &= d''(a_v - a_u' + a'b - ab' + a''c - a'c') - d'(a_v' - a_u'' + a'b' - ab'' + a''c' - a'c''),\\
\frac{1}{\Delta^2}\beta' &= d''(b_v - b_u' - a' + b''c - b'c') - d'(b_v' - b_u'' - a'' + b'^2 - bb'' + b''c' - b'c''),\\
\frac{1}{\Delta^2}\gamma' &= d''(c_v - c_u' + a + bc' - b'c + cc'' - c'^2) - d'(c_v' - c_u'' + a' + b'c' - b''c),\\
\frac{1}{\Delta^2}\delta' &= d''(d_v - d_u' + bd' - b'd + cd'' - c'd') - d'(d_v' - d_u'' + b'd' - b''d + c'd'' - c''d').
\end{aligned}
\tag{16}
$$

As a consequence of these equations (15), equations (12) and (13), from which they were derived, show that the six expressions for the four third derivatives of y will be consistent. In order that the same thing may be true of all of the derivatives of order higher than three, it is necessary and sufficient that equations (15) themselves may be consistent, i. e., that

$$
\frac{\partial w_u}{\partial v} = \frac{\partial w_v}{\partial u}. \tag{17}
$$

But it has already been remarked that there can be no relation of form (11) with non-vanishing coefficients. The condition (17) however would give rise to such a relation unless the coefficients of y, y_u, y_v and w on the left member were equal respectively to the corresponding coefficients on the right member of that equation. We thus obtain the following four relations:

$$
\begin{aligned}
\alpha_v + a'\beta + a''\gamma + \alpha'\delta &= \alpha_u' + a\beta' + a'\gamma' + \alpha\delta',\\
\beta_v + b'\beta + b''\gamma + \beta'\delta &= \beta_u' + \alpha' + b\beta' + b'\gamma' + \beta\delta',\\
\gamma_v + \alpha + c'\beta + c''\gamma + \gamma'\delta &= \gamma_u' + c\beta' + c'\gamma' + \gamma\delta',\\
\delta_v + d'\beta + d''\gamma &= \delta_u' + d\beta' + d'\gamma',
\end{aligned}
\tag{18}
$$

the integrability conditions of system (6).

The last of these four conditions may be transformed as follows. It may be written

$$
\delta_v - \delta_u' + d'\beta - d\beta' + d''\gamma - d'\gamma' = 0.
$$

But

$$
\begin{aligned}
d'\beta - d\beta' &= b_v - b_u' - a' + b''c - b'c',\\
d''\gamma - d'\gamma' &= c_v' - c_u'' + a' + b'c' - b''c,
\end{aligned}
$$

so that the condition becomes

$$\delta_v + b_v + c'_v = \delta'_u + b'_u + c''_u. \tag{19}$$

For the sake of brevity put

$$\begin{aligned} \mu &= bd' - b'd + cd'' - c'd', \\ \nu &= b'd' - b''d + c'd'' - c''d'. \end{aligned} \tag{20}$$

Then, according to (16),

$$\begin{aligned} \delta &= \Delta^2[d'(d_v - d'_u) - d(d'_v - d''_u) + d'\mu - d\nu], \\ \delta' &= \Delta^2[d''(d_v - d'_u) - d'(d'_v - d''_u) + d''\mu - d'\nu]. \end{aligned}$$

But from equations (9) we see that

$$\mu = \frac{1}{\Delta}(q'^2 s - q^2 s' - p'q' t + pqt'),$$

$$\nu = \frac{1}{\Delta}(-p'q' s + pqs' + p'^2 t - p^2 t'),$$

so that

$$\begin{aligned} d'\mu - d\nu &= \frac{1}{\Delta^2}(b + c'), \\ d''\mu - d'\nu &= \frac{1}{\Delta^2}(b' + c''), \end{aligned} \tag{21}$$

whence

$$\begin{aligned} \delta &= \Delta^2[d'(d_v - d'_u) - d(d'_v - d''_u)] + b + c', \\ \delta' &= \Delta^2[d''(d_v - d'_u) - d'(d'_v - d''_u)] + b' + c''. \end{aligned} \tag{22}$$

With the aid of these expressions, it will be seen that the last of the four integrability conditions (18) may be satisfied in the most general way by putting

$$\begin{aligned} \Delta^2[d'(d_v - d'_u) - d(d'_v - d''_u)] + 2b + 2c' &= \frac{\partial \epsilon'}{\partial u}, \\ \Delta^2[d''(d_v - d'_u) - d'(d'_v - d''_u)] + 2b' + 2c'' &= \frac{\partial \epsilon'}{\partial v}, \end{aligned} \tag{23}$$

where ϵ' is an arbitrary function of u and v. We may use these equations to express b and c'' in terms of ϵ' and the other coefficients of system (1).

In will be advantageous to introduce

$$D = \Delta d, \qquad D' = \Delta d', \qquad D'' = \Delta d'', \tag{24}$$

in place of d, d', d''. Then

$$D'^2 - DD'' = \Delta^2(d'^2 - dd'') = 1, \tag{25}$$

and the conditions (23) become

$$\begin{aligned} D'(D_v - D'_u) - D(D'_v - D''_u) + 2(b + c') &= \epsilon_u, \\ D''(D_v - D'_u) - D'(D'_v - D''_u) + 2(b' + c'') &= \epsilon_v, \end{aligned} \tag{26}$$

where

$$\epsilon = \epsilon' - \log \Delta$$

may be any function of u and v.

§ 3. *The seminvariants.*

Let us consider the system (6), which is in its normal form. If we multiply the members of both equations by arbitrary functions ω^2 and ω'^2 of u and v, the resulting system is again in the normal form. Moreover, if the two equations of the system be interchanged, the normal form is not disturbed. The combination of these two operations gives rise to the most general system of equations equivalent to (6) and still in the normal form (cf. § 1). Only those combinations of the coefficients of (6) which remain unchanged as a result of these operations are of interest to us, because they alone represent quantities determined by the system which are identical for all equivalent systems. In other words, they are invariants of the system (1) under the transformation which consists in replacing it by any equivalent system (2). The twelve quantities, $a, b, c, D, a', b', c', D', a'', b'', c'', D''$, the *virtual coefficients* of the system, are invariants of this kind. It is for this reason that we have introduced D, D', D'' in place of d, d', d'' at the end of the preceding paragraph. These twelve quantities, however, are not independent. In fact we find the following relations between them

$$\begin{aligned} aD'' - 2a'D' + a''D &= 0, \\ bD'' - 2b'D' + b''D &= 0, \\ cD'' - 2c'D' + c''D &= 0, \\ D'^2 - DD'' &= 1, \end{aligned} \tag{27}$$

from which others may be derived, of which we shall mention only the following:

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0. \tag{28}$$

Since we may substitute equations (10) for (6), all invariants of system (6) may be expressed in terms of the twelve quantities $a, b, c, D, \cdots, D''$. Moreover, only eight of these are independent owing to the four independent relations (27), which is in harmony with the fact that system (6) involves only the eight ratios of the ten functions $h, k, h', k', \mathrm{r}, \mathrm{s}, \mathrm{t}, \mathrm{r}', \mathrm{s}', \mathrm{t}'$; but we shall retain all twelve, for the sake of symmetry.

Let us now put in (6)

$$y = \lambda \bar{y}, \tag{29}$$

and afterward divide both members of both equations by λ, where λ is an arbitrary function of u and v. The functions of the quantities, a, b, c, D, etc., and of their derivatives which remain unaltered by this transformation shall be called the *seminvariants* of system (6). The *semicovariants* have the same invariant property, but involve y and its derivatives besides the coefficients of the system.

We have from (29)

$$
\begin{aligned}
&y_u = \lambda \bar{y}_u + \lambda_u y, \qquad y_v = \lambda \bar{y}_v + \lambda_v \bar{y},\\
&y_{uu} = \lambda \bar{y}_{uu} + 2\lambda_u \bar{y}_u + \lambda_{uu} \bar{y},\\
&y_{uv} = \lambda \bar{y}_{uv} + \lambda_v \bar{y}_u + \lambda_u \bar{y}_v + \lambda_{uv} \bar{y},\\
&y_{vv} = \lambda \bar{y}_{vv} + 2\lambda_v \bar{y}_v + \lambda_{vv} \bar{y}.
\end{aligned} \tag{30}
$$

Substitute these values in (6), divide by λ, and denote the coefficients of the resulting system of equations by $\bar{h}$, $\bar{k}$, $\bar{\mathrm{r}}$, $\bar{\mathrm{s}}$, $\bar{\mathrm{t}}$, etc. We find

$$
\begin{aligned}
&\bar{h} = \pm h, \qquad \bar{k} = \pm k,\\
&\bar{\mathrm{r}} = \mathrm{r} + \frac{\lambda_u}{\lambda}\mathrm{s} + \frac{\lambda_v}{\lambda}\mathrm{t} - \frac{\lambda_{uu}}{\lambda}h^2 - 2\,\frac{\lambda_{uv}}{\lambda}hk - \frac{\lambda_{vv}}{\lambda}k^2,\\
&\bar{\mathrm{s}} = \mathrm{s} - 2h\left(\frac{\lambda_u}{\lambda}h + \frac{\lambda_v}{\lambda}k\right),\\
&\bar{\mathrm{t}} = \mathrm{t} - 2k\left(\frac{\lambda_u}{\lambda}h + \frac{\lambda_v}{\lambda}k\right),
\end{aligned}
$$

the equations of transformation for h', k', r', s', t' being of precisely the same form. Moreover, in the first two equations, either both of the upper signs or both of the lower signs must be taken.

We find consequently:

$$
\begin{aligned}
&\bar{a} = a + \frac{\lambda_u}{\lambda}b + \frac{\lambda_v}{\lambda}c - \frac{\lambda_{uu}}{\lambda}T + \frac{\lambda_{uv}}{\lambda}DD' - \frac{1}{2}\frac{\lambda_{vv}}{\lambda}D^2,\\
&\bar{a}' = a' + \frac{\lambda_u}{\lambda}b' + \frac{\lambda_v}{\lambda}c' - \frac{1}{2}\frac{\lambda_{uu}}{\lambda}D'D'' + \frac{\lambda_{uv}}{\lambda}DD'' - \frac{1}{2}\frac{\lambda_{vv}}{\lambda}DD',\\
&\bar{a}'' = a'' + \frac{\lambda_u}{\lambda}b'' + \frac{\lambda_v}{\lambda}c'' - \frac{1}{2}\frac{\lambda_{uu}}{\lambda}D''^2 + \frac{\lambda_{uv}}{\lambda}D'D'' - \frac{\lambda_{vv}}{\lambda}T,\\
&\bar{b} = b - 2\frac{\lambda_u}{\lambda}T + \frac{\lambda_v}{\lambda}DD', \qquad \bar{c} = c + \frac{\lambda_u}{\lambda}DD' - \frac{\lambda_v}{\lambda}D^2,\\
&\bar{b}' = b' - \frac{\lambda_u}{\lambda}D'D'' + \frac{\lambda_v}{\lambda}DD'', \qquad \bar{c}' = c' + \frac{\lambda_u}{\lambda}DD'' - \frac{\lambda_v}{\lambda}DD',\\
&\bar{b}'' = b'' - \frac{\lambda_u}{\lambda}D''^2 + \frac{\lambda_v}{\lambda}D'D'', \qquad \bar{c}'' = c'' + \frac{\lambda_u}{\lambda}D'D'' - 2\frac{\lambda_v}{\lambda}T,\\
&\bar{D} = D, \qquad \bar{D}' = D', \qquad \bar{D}'' = D'',
\end{aligned} \tag{31}
$$

where

$$T = D^2 - \tfrac{1}{2}DD''. \tag{32}$$

Therefore D, D', D'' are seminvariants. Moreover the system of equations (27) remains invariant under this transformation, although the left member of the first of these equations is not by itself a seminvariant.

Since we are supposing the integrability conditions to be satisfied, the coefficients of our system will satisfy equations (26) where ϵ is some function of u and v. Let $\bar{\epsilon}$ denote the value of this function after the transformation. We find, from (26) and (31),

$$\bar{\epsilon}_u = \epsilon_u - 4\frac{\lambda_u}{\lambda}, \qquad \bar{\epsilon}_v = \epsilon_v - 4\frac{\lambda_v}{\lambda}.$$

Consequently $\bar{\epsilon}_u$ and $\bar{\epsilon}_v$ may be reduced to zero in the most general way by putting

$$\lambda = ke^{\frac{1}{4}\epsilon},$$

where k is an arbitrary constant. We shall then have

$$\frac{\lambda_u}{\lambda} = \tfrac{1}{4}\epsilon_u, \qquad \frac{\lambda_v}{\lambda} = \tfrac{1}{4}\epsilon_v,$$

$$\frac{\lambda_{uu}}{\lambda} = \tfrac{1}{4}\epsilon_{uu} + \tfrac{1}{16}\epsilon_u^2, \qquad \frac{\lambda_{uv}}{\lambda} = \tfrac{1}{4}\epsilon_{uv} + \tfrac{1}{16}\epsilon_u\epsilon_v, \qquad \frac{\lambda_{vv}}{\lambda} = \tfrac{1}{4}\epsilon_{vv} + \tfrac{1}{16}\epsilon_v^2. \tag{33}$$

The substitution

$$y = ke^{\frac{1}{4}\epsilon}\bar{y}$$

will reduce system (6) to another one of the same form. The quantities A, B, C, D, etc., which correspond to a, b, c, D, etc. in this new system, are obtained by substituting the values (33) for λ_u/λ, λ_v/λ, etc., in equations (31). These quantities satisfy the relations

$$\begin{aligned} D'(D_v - D'_u) - D(D'_v - D''_u) + 2(B + C') &= 0, \\ D''(D_v - D'_u) - D'(D'_v - D''_u) + 2(B' + C''') &= 0, \end{aligned} \tag{34}$$

obtained from (26) on account of the vanishing of $\bar{\epsilon}_u$ and $\bar{\epsilon}_v$.

We have called the quantities a, b, c, D, $\cdots$ the *virtual coefficients* of system (6). We have seen that there exists for every such system a uniquely determined transformed system whose virtual coefficients satisfy the conditions (34). This transformed system shall be said to be in its *semi-canonical form.*

The virtual coefficients of the semi-canonical form, A, B, C, D, *etc., are seminvariants of system* (6). They are the values of $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{D}$, etc., obtained from (31) by substituting for λ_u/λ, λ_v/λ, etc., the values (33).

The seminvariance of these quantities is a consequence of the uniqueness of the semi-canonical form, and may moreover be tested directly by means of equations (31).

As a consequence of (27) *the seminvariants satisfy the relations*

$$\begin{aligned} AD'' - 2A'D' + A''D &= 0, \\ BD'' - 2B'D' + B''D &= 0, \\ CD'' - 2C'D' + C''D &= 0, \\ D'^2 - DD'' &= 1, \end{aligned} \tag{35}$$

as well as (34), *so that only seven of them are independent. The complete system of seminvariants obviously consists of these quantities and their derivatives with respect to the two independent variables u and v.*

In fact, every seminvariant S is a function of the virtual coefficients of the system considered and of their derivatives, which does not change its value when any transformation of the form

$$y = \lambda \bar{y}$$

is made. But if the system be reduced to the semi-canonical form, every such function becomes a function of *its* virtual coefficients alone, i. e., of A, B, C, D, etc., and of their derivatives. Since S does not change its value as a result of this reduction, S is therefore a function of these variables only.

§ 4. *The invariants.*

We now proceed to investigate the effect of a transformation of the independent variables upon the seminvariants. Let new variables be introduced by means of the equations

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v), \tag{36}$$

where ϕ and ψ denote arbitrary functions of u and v. We find

$$\begin{aligned} y_u &= y_{\bar{u}}\phi_u + y_{\bar{v}}\psi_u, \qquad y_v = y_{\bar{u}}\phi_v + y_{\bar{v}}\psi_v, \\ y_{uu} &= y_{\bar{u}\bar{u}}\phi_u^2 + 2y_{\bar{u}\bar{v}}\phi_u\psi_u + y_{\bar{v}\bar{v}}\psi_u^2 + y_{\bar{u}}\phi_{uu} + y_{\bar{v}}\psi_{uu}, \\ y_{uv} &= y_{\bar{u}\bar{u}}\phi_u\phi_v + y_{\bar{u}\bar{v}}(\phi_u\psi_v + \phi_v\psi_u) + y_{\bar{v}\bar{v}}\psi_u\psi_v + y_{\bar{u}}\phi_{uv} + y_{\bar{v}}\psi_{uv}, \\ y_{vv} &= y_{\bar{u}\bar{u}}\phi_v^2 + 2y_{\bar{v}\bar{u}}\phi_v\psi_v + y_{\bar{v}\bar{v}}\psi_v^2 + y_{\bar{u}}\phi_{vv} + y_{\bar{v}}\psi_{vv}. \end{aligned} \tag{37}$$

Substitute these values in system (6) and denote the coefficients of the transformed system by $\bar{h}$, $\bar{k}$, $\bar{\mathrm{r}}$, $\bar{\mathrm{s}}$, $\bar{\mathrm{t}}$, etc. We shall find

$$\begin{aligned} \bar{h} &= \pm(\phi_u h + \phi_v k), \qquad \bar{k} = \pm(\psi_u h + \psi_v k), \qquad \bar{\mathrm{r}} = \mathrm{r}, \\ \bar{\mathrm{s}} &= \phi_u \mathrm{s} + \phi_v \mathrm{t} - \phi_{uu}h^2 - 2\phi_{uv}hk - \phi_{vv}k^2, \\ \bar{\mathrm{t}} &= \psi_u \mathrm{s} + \psi_v \mathrm{t} - \psi_{uu}h^2 - 2\psi_{uv}hk - \psi_{vv}k^2, \end{aligned} \tag{38}$$

and similar expressions for $\bar{K}$, $\bar{E}$, etc. The two upper or the two lower signs

may be chosen, in (38), at will; for the virtual coefficients and consequently the seminvariants contain only such combinations of h, k, h', k' as are unaffected by the sign chosen.

Introduce infinitesimal transformations by putting

$$(39)\qquad \bar{u} = \phi(u, v) = u + \kappa(u, v)\delta t, \qquad \bar{v} = \psi(u, v) = v + \lambda(u, v)\delta t,$$

where δt is an infinitesimal. We find the following infinitesimal transformations for h, k, etc.:

$$\delta h = \pm(\kappa_u h + \kappa_v k)\delta t, \qquad \delta k = \pm(\lambda_u h + \lambda_v k)\delta t, \qquad \delta r = 0,$$

$$(40)\qquad \delta s = (\kappa_u s + \kappa_v t - \kappa_{uu} h^2 - 2\kappa_{uv} hk - \kappa_{vv} k^2)\delta t,$$

$$\delta t = (\lambda_u s + \lambda_v t - \lambda_{uu} h^2 - 2\lambda_{uv} hk - \lambda_{vv} k^2)\delta t.$$

The equations for $\delta h', \cdots \delta t'$ are of exactly the same form. Making use of the equations (9) and (24) which define a, b, c, D, etc., we find:

$$\begin{aligned}
\delta a &= -(2a\kappa_u + 2a'\lambda_u)\delta t,\\
\delta a' &= -(a'\kappa_u + a\kappa_v + a''\lambda_u + a'\lambda_v)\delta t,\\
\delta a'' &= -(2a'\kappa_v + 2a''\lambda_v)\delta t,\\
\delta b &= [-b\kappa_u + c\kappa_v - 2b'\lambda_u - T\kappa_{uu} + DD'\kappa_{uv} - \tfrac{1}{2}D^2\kappa_{vv}]\delta t,\\
\delta b' &= [(c' - b)\kappa_v - b''\lambda_u - b'\lambda_v - \tfrac{1}{2}D'D''\kappa_{uu} + DD''\kappa_{uv} - \tfrac{1}{2}DD'\kappa_{vv}]\delta t,\\
\delta b'' &= [b''\kappa_u + (c'' - 2b')\kappa_v - 2b''\lambda_v - \tfrac{1}{2}D''^2\kappa_{uu} + D'D''\kappa_{uv} - T\kappa_{vv}]\delta t,\\
\delta c &= [-2c\kappa_u + (b - 2c')\lambda_u + c\lambda_v - T\lambda_{uu} + DD'\lambda_{uv} - \tfrac{1}{2}D^2\lambda_{vv}]\delta t,\\
\delta c' &= [-c'\kappa_u - c\kappa_v + (b' - c'')\lambda_u - \tfrac{1}{2}D'D''\lambda_{uu} + DD''\lambda_{uv} - \tfrac{1}{2}DD'\lambda_{vv}]\delta t,\\
\delta c'' &= [-2c'\kappa_v + b''\lambda_u - c''\lambda_v - \tfrac{1}{2}D''^2\lambda_{uu} + D'D''\lambda_{uv} - T\lambda_{vv}]\delta t,\\
\delta D &= (-D\kappa_u - 2D'\lambda_u + D\lambda_v)\delta t,\\
\delta D' &= (-D\kappa_v - D''\lambda_u)\delta t,\\
\delta D'' &= (D''\kappa_u - 2D'\kappa_v - D''\lambda_v)\delta t.
\end{aligned} \qquad (41)$$

The infinitesimal transformations of the derivatives of these quantities may be obtained by noticing that, for any function Φ, the following equations hold:

$$\begin{aligned}
\delta\Phi_u &= \frac{\partial}{\partial u}(\delta\Phi) - (\kappa_u\Phi_u + \lambda_u\Phi_v)\delta t,\\
\delta\Phi_v &= \frac{\partial}{\partial v}(\delta\Phi) - (\kappa_v\Phi_u + \lambda_v\Phi_v)\delta t.
\end{aligned} \qquad (42)$$

Thus we find in particular:

$$
\begin{aligned}
\delta D_u &= [-2D_u\kappa_u - (2D'_u + D_v)\lambda_u + D_u\lambda_v - D\kappa_{uu} - 2D'\lambda_{uu} + D\lambda_{uv}]\delta t,\\
\delta D_v &= [D_v\kappa_u - D_u\kappa_v - 2D'_u\lambda_u - D\kappa_{uv} - 2D'\lambda_{uv} + D\lambda_{vv}]\delta t,\\
\delta D'_u &= [-D'_u\kappa_u - D_u\kappa_v - (D''_u + D'_v)\lambda_u - D\kappa_{uv} - D'\lambda_{uu}]\,\delta t,\\
\delta D'_v &= [-(D_v + D'_u)\kappa_v - D''_v\lambda_u - D'_v\lambda_v - D\kappa_{vv} - D'\lambda_{uv}]\,\delta t,\\
\delta D''_u &= [-2D'_u\kappa_v - D''_v\lambda_u - D''_u\lambda_v + D''\kappa_{uu} - 2D'\kappa_{uv} - D''\lambda_{uv}]\,\delta t,\\
\delta D''_v &= [D''_v\kappa_u - (2D'_v + D''_u)\kappa_v - 2D''_v\lambda_v + D''\kappa_{uv} - 2D'\kappa_{vv} - D''\lambda_{vv}]\,\delta t.
\end{aligned}
\tag{43}
$$

From (26) we now find

$$
\begin{aligned}
\delta\epsilon_u &= -(\epsilon_u\kappa_u + \epsilon_v\lambda_u + 2\kappa_{uu} + 2\lambda_{uv})\,\delta t,\\
\delta\epsilon_v &= -(\epsilon_u\kappa_v + \epsilon_v\lambda_v + 2\kappa_{uv} + 2\lambda_{vv})\,\delta t,
\end{aligned}
\tag{44}
$$

whence

$$
\begin{aligned}
\delta\epsilon_{uu} &= -(2\epsilon_{uu}\kappa_u + 2\epsilon_{uv}\lambda_u + \epsilon_u\kappa_{uu} + \epsilon_v\lambda_{uu} + 2\kappa_{uuu} + 2\lambda_{uuv})\,\delta t,\\
\delta\epsilon_{uv} &= -(\epsilon_{uv}\kappa_u + \epsilon_{uu}\kappa_v + \epsilon_{vv}\lambda_u + \epsilon_{uv}\lambda_v + \epsilon_u\kappa_{uv} + \epsilon_v\lambda_{uv} + 2\kappa_{uuv} + 2\lambda_{uvv})\,\delta t,\\
\delta\epsilon_{vv} &= -(2\epsilon_{uv}\kappa_v + 2\epsilon_{vv}\lambda_v + \epsilon_u\kappa_{vv} + \epsilon_v\lambda_{vv} + 2\kappa_{uvv} + 2\lambda_{vvv})\,\delta t.
\end{aligned}
\tag{45}
$$

The quantities ϵ_u, ϵ_v, etc., enter into the composition of the seminvariants. Making use of equations (31), (33), (41), (44), (45), we obtain following expressions for their infinitesimal transformations:

$$
\begin{aligned}
\delta A &= [-2A\kappa_u - 2A'\lambda_u - \tfrac{1}{2}B(\kappa_{uu} + \lambda_{uv}) - \tfrac{1}{2}C'(\kappa_{uv} + \lambda_{vv})\\
&\quad + \tfrac{1}{2}T(\kappa_{uuu} + \lambda_{uuv}) - \tfrac{1}{2}DD'(\kappa_{uuv} + \lambda_{uvv}) + \tfrac{1}{4}D^2(\kappa_{uvv} + \lambda_{vvv})]\,\delta t,\\
\delta A' &= [-A'\kappa_u - A\kappa_v - A''\lambda_u - A'\lambda_v - \tfrac{1}{2}B'(\kappa_{uu} + \lambda_{uv}) - \tfrac{1}{2}C'(\kappa_{uv} + \lambda_{vv})\\
&\quad + \tfrac{1}{4}D'D''(\kappa_{uuu} + \lambda_{uuv}) - \tfrac{1}{2}DD''(\kappa_{uuv} + \lambda_{uvv}) + \tfrac{1}{4}DD'(\kappa_{uvv} + \lambda_{vvv})]\,\delta t,\\
\delta A'' &= [-2A'\kappa_v - 2A''\lambda_v - \tfrac{1}{2}B''(\kappa_{uu} + \lambda_{uv}) - \tfrac{1}{2}C''(\kappa_{uv} + \lambda_{vv})\\
&\quad + \tfrac{1}{4}D''^2(\kappa_{uuu} + \lambda_{uuv}) - \tfrac{1}{2}D'D''(\kappa_{uuv} + \lambda_{uvv}) + \tfrac{1}{2}T(\kappa_{uvv} + \lambda_{vvv})]\,\delta t,
\end{aligned}
\tag{46}
$$

and further,

$$
\begin{aligned}
\delta B &= [-B\kappa_u + C\kappa_v - 2B'\lambda_u + \tfrac{1}{2}DD'(\kappa_{uv} - \lambda_{vv}) - \tfrac{1}{2}D^2\kappa_{vv} + T\lambda_{uv}]\,\delta t,\\
\delta B' &= [(C' - B)\kappa_v - B''\lambda_u - B'\lambda_v + \tfrac{1}{2}DD''(\kappa_{uv} - \lambda_{vv}) - \tfrac{1}{2}DD'\kappa_{vv} + \tfrac{1}{2}D'D''\lambda_{uv}]\delta t,\\
\delta B'' &= [B''\kappa_u + (C'' - 2B')\kappa_v - 2B''\lambda_v + \tfrac{1}{2}D'D''(\kappa_{uv} - \lambda_{vv}) - T\kappa_{vv} + \tfrac{1}{2}D''^2\lambda_{uv}]\delta t,\\
\delta C &= [-2C\kappa_u + (B - 2C')\lambda_u + C\lambda_v - \tfrac{1}{2}DD'(\kappa_{uu} - \lambda_{uv}) + \tfrac{1}{2}D^2\kappa_{uv} - T\lambda_{uu}]\,\delta t,\\
\delta C' &= [-C'\kappa_u - C\kappa_v + (B' - C'')\lambda_u - \tfrac{1}{2}DD''(\kappa_{uu} - \lambda_{uv})\\
&\quad + \tfrac{1}{2}DD'\kappa_{uv} - \tfrac{1}{2}D'D''\lambda_{uu}]\,\delta t,\\
\delta C'' &= [-2C'\kappa_v + B''\lambda_u - C''\lambda_v - \tfrac{1}{2}D'D''(\kappa_{uu} - \lambda_{uv}) + T\kappa_{uv} - \tfrac{1}{2}D''^2\lambda_{uu}]\,\delta t.
\end{aligned}
\tag{47}
$$

Let us put

$$
\begin{aligned}
M &= D'D_v - DD'_v, & N &= D'D_u - DD'_u,\\
M' &= D''D_v - DD''_v, & N' &= D''D_u - DD''_u,\\
M'' &= D''D'_v - D'D''_v, & N'' &= D''D'_u - D'D''_u,
\end{aligned}
\tag{48}
$$

Then will

$$
\begin{aligned}
\delta M &= [-M\kappa_u - N\kappa_v - M'\lambda_u - DD'(\kappa_{uv} - \lambda_{vv}) + D^2\kappa_{vv} - 2T\lambda_{uv}]\delta t,\\
\delta M' &= [-(N' + 2M)\kappa_v - 2M''\lambda_u - M'\lambda_v - 2DD'(\kappa_{uv} - \lambda_{vv})\\
&\qquad + 2DD'\kappa_{vv} - 2D'D'\lambda_{uv}]\,\delta t,\\
\delta M'' &= [M''\kappa_u - (M' + N'')\kappa_v - 2M''\lambda_v - D'D''(\kappa_{uv} - \lambda''_{vv}) + 2T\kappa_{vv} - D''^2\lambda_{uv}]\delta t,\\
\delta N &= [-2N\kappa_u - (M + N')\lambda_u + N\lambda_v - DD'(\kappa_{uu} - \lambda_{uv}) + D^2\kappa_{uv} - 2T\lambda_{uu}]\delta t,\\
\delta N' &= [-N'\kappa_u - 2N\kappa_v - (M' + 2N'')\lambda_u - 2DD''(\kappa_{uu} - \lambda_{uv})\\
&\qquad + 2DD'\kappa_{uv} - 2D'D''\lambda_{uu}]\,\delta t,\\
\delta N'' &= [-N'\kappa_v - M''\lambda_u - N''\lambda_v - D'D''(\kappa_{uu} - \lambda_{uv}) + 2T\kappa_{uv} - D''^2\lambda_{uu}]\delta t.
\end{aligned}
\tag{49}
$$

If we put now

$$
\begin{aligned}
\mathfrak{B} &= B + \tfrac{1}{2}M, & \mathfrak{C} &= C - \tfrac{1}{2}N,\\
\mathfrak{B}' &= B' + \tfrac{1}{4}M', & \mathfrak{C}' &= C' - \tfrac{1}{4}N',\\
\mathfrak{B}'' &= B'' + \tfrac{1}{2}M'', & \mathfrak{C}'' &= C'' - \tfrac{1}{2}N'',
\end{aligned}
\tag{50}
$$

we shall therefore find

$$
\begin{aligned}
\delta\mathfrak{B} &= [-\mathfrak{B}\kappa_u + \mathfrak{C}\kappa_v - 2\mathfrak{B}'\lambda_u]\delta t, & \delta\mathfrak{C} &= [-2\mathfrak{C}\kappa_u + (\mathfrak{B} - 2\mathfrak{C}')\lambda_u + \mathfrak{C}\lambda_v]\delta t,\\
\delta\mathfrak{B}' &= [(\mathfrak{C}' - \mathfrak{B})\kappa_v - \mathfrak{B}''\lambda_u - \mathfrak{B}'\lambda_v]\delta t, & \delta\mathfrak{C}' &= [-\mathfrak{C}'\kappa_u - \mathfrak{C}\kappa_v + (\mathfrak{B}' - \mathfrak{C}'')\lambda_u]\delta t,\\
\delta\mathfrak{B}'' &= [\mathfrak{B}''\kappa_u + (\mathfrak{C}'' - 2\mathfrak{B}')\kappa_v - 2\mathfrak{B}''\lambda_v]\delta t, & \delta\mathfrak{C}'' &= [-2\mathfrak{C}'\kappa_v + \mathfrak{B}''\lambda_u - \mathfrak{C}''\lambda_v]\delta t.
\end{aligned}
\tag{51}
$$

From (34), (35), (48) and (50) we find the following relations:

$$
\begin{aligned}
&\mathfrak{B}D'' - 2\mathfrak{B}'D' + \mathfrak{B}''D = 0, \qquad \mathfrak{C}D'' - 2\mathfrak{C}'D' + \mathfrak{C}''D = 0,\\
&\mathfrak{B} + \mathfrak{C}' = 0, \qquad \mathfrak{B}' + \mathfrak{C}'' = 0, \qquad D'^2 - DD'' = 1,
\end{aligned}
\tag{52}
$$

which system of equations remains invariant under all of the transformations here considered.

By means of the relations (52), five of the nine variables $\mathfrak{B}$, $\mathfrak{B}'$, $\mathfrak{B}''$, $\mathfrak{C}$, $\mathfrak{C}'$, $\mathfrak{C}''$, D, D', D'' may be expressed in terms of the other four. For instance, if

$$DD'' - 4D'^2 \neq 0,$$

we shall find

$$\mathfrak{C}' = -\mathfrak{B} = \frac{\mathfrak{B}''D' - 2\mathfrak{C}D'D''}{DD'' - 4D'^2},$$

$$\mathfrak{B}' = -\mathfrak{C}'' = \frac{-2\mathfrak{B}''DD' + \mathfrak{C}D''^2}{DD'' - 4D'^2},$$

$$D' = \sqrt{1 + DD''}.$$

Any invariant, depending only upon these nine seminvariants, therefore becomes a function of D, D'', $\mathfrak{B}''$ and $\mathfrak{C}$ alone. From (41) and (51) we see that every absolute invariant of this kind must satisfy a complete system of four linear homogeneous partial differential equations of the first order with four independent variables. These equations are found to be independent, a fact which may also be expressed by saying that the determinant Δ_4 of the fourth order made up of the sixteen coefficients of these four equations is not identically equal to zero. Therefore *no such absolute invariant exists.* To be sure, the curves $u = \text{const.}$ and $v = \text{const.}$ upon the integral surface might be such as to make $DD'' - 4D'^2$ vanish, so as to destroy the validity of this argument. But this condition

$$DD'' - 4D'^2 = 0$$

is not invariant, and may always be avoided by a transformation of the independent variables. Such a transformation cannot, however, affect the existence or non-existence of an invariant. We have shown therefore that there exists no absolute invariant which involves only the above nine seminvariants. Therefore there can be at most one *relative* invariant which is a function of these nine variables alone. Such a one actually exists and, owing to the relations (52), may be written in various forms. We proceed to obtain it.

Let

(53) $$P = \mathfrak{B}'\mathfrak{C}'' - \mathfrak{B}''\mathfrak{C}', \qquad R = \mathfrak{B}\mathfrak{C}'' - \mathfrak{B}''\mathfrak{C}, \qquad S = \mathfrak{B}\mathfrak{C}' - \mathfrak{B}'\mathfrak{C}.$$

Then wil

(54) $$\delta P = -(R\kappa_v + 2P\lambda_v)\delta t,$$

$$\delta R = -(R\kappa_u + 2S\kappa_v + 2P\lambda_u + R\lambda_v)\delta t,$$

$$\delta S = -(2S\kappa_u + R\lambda_u)\delta t.$$

If these equations be combined with the last three equations of system (41), which give the infinitesimal transformations of D, D', D'', it will be seen that *θ is a relative invariant, where*

(55) $$\theta = \begin{vmatrix} \mathfrak{B} & \mathfrak{B}' & \mathfrak{B}'' \\ \mathfrak{C} & \mathfrak{C}' & \mathfrak{C}'' \\ D & D' & D'' \end{vmatrix} = PD - RD' + SD''.$$

In fact we find

$$\delta\theta = -(\kappa_u + \lambda_v)\theta\delta t. \tag{56}$$

We wish to determine next those invariants which involve the first derivatives of the quantities $\mathfrak{B}$, $\mathfrak{C}$ and D, as well as these quantities themselves. We shall see that they must be solutions of a complete system of ten partial differential equations with twelve independent variables. There must exist, consequently, two absolute, or three relative invariants of this kind including θ; therefore there will be two new relative invariants to be determined.

The expressions for δD_u, etc., have already been deduced [cf. eq. (43)]. From (51) and (42) we find the following formulae:

$$\begin{aligned}
\delta\mathfrak{B}_u &= [-2\mathfrak{B}_u\kappa_u + \mathfrak{C}_u\kappa_v - (2\mathfrak{B}'_u + \mathfrak{B}_v)\lambda_u - \mathfrak{B}\kappa_{uu} + \mathfrak{C}\kappa_{uv} - 2\mathfrak{B}'\lambda_{uu}]\delta t,\\
\delta\mathfrak{B}_v &= [-\mathfrak{B}_v\kappa_u + (\mathfrak{C}_v - \mathfrak{B}_u)\kappa_v - 2\mathfrak{B}'_v\lambda_u - \mathfrak{B}_v\lambda_v - \mathfrak{B}\kappa_{uv} + \mathfrak{C}\kappa_{vv} - 2\mathfrak{B}'\lambda_{uv}]\delta t,\\
\delta\mathfrak{B}''_u &= [(\mathfrak{C}''_u - 2\mathfrak{B}'_u)\kappa_v - \mathfrak{B}''_v\lambda_u - 2\mathfrak{B}''_u\lambda_v + \mathfrak{B}''\kappa_{uu} + (\mathfrak{C}'' - 2\mathfrak{B}')\kappa_{uv} - 2\mathfrak{B}''\lambda_{uv}]\delta t,\\
\delta\mathfrak{B}''_v &= [\mathfrak{B}''_v\kappa_u + (\mathfrak{C}''_v - 2\mathfrak{B}'_v - \mathfrak{B}''_u)\kappa_v - 3\mathfrak{B}''_v\lambda_v + \mathfrak{B}''\kappa_{uv} + (\mathfrak{C}'' - 2\mathfrak{B}')\kappa_{vv} - 2\mathfrak{B}''\lambda_{vv}]\delta t,\\
\delta\mathfrak{C}_u &= [-3\mathfrak{C}_u\kappa_u + (\mathfrak{B}_u - 2\mathfrak{C}'_u - \mathfrak{C}_v)\lambda_u + \mathfrak{C}_u\lambda_v - 2\mathfrak{C}\kappa_{uu} + (\mathfrak{B} - 2\mathfrak{C}')\lambda_{uu} + \mathfrak{C}\lambda_{uv}]\delta t,\\
\partial\mathfrak{C}_v &= [-2\mathfrak{C}_v\kappa_u - \mathfrak{C}_u\kappa_v + (\mathfrak{B}_v - 2\mathfrak{C}'_v)\lambda_u - 2\mathfrak{C}\kappa_{uv} + (\mathfrak{B} - 2\mathfrak{C}')\lambda_{uv} + \mathfrak{C}\lambda_{vv}]\delta t,\\
\delta\mathfrak{C}''_u &= [-\mathfrak{C}''_u\kappa_u - 2\mathfrak{C}'_u\kappa_v + (\mathfrak{B}''_u - \mathfrak{C}''_v)\lambda_u - \mathfrak{C}''_u\lambda_v - 2\mathfrak{C}'\kappa_{uv} + \mathfrak{B}''\lambda_{uu} - \mathfrak{C}''\lambda_{uv}]\delta t,\\
\delta\mathfrak{C}''_v &= [-(2\mathfrak{C}'_v + \mathfrak{C}''_u)\kappa_v + \mathfrak{B}''_v\lambda_u - 2\mathfrak{C}''_v\lambda_v - 2\mathfrak{C}'\kappa_{vv} + \mathfrak{B}''\lambda_{uv} - \mathfrak{C}''\lambda_{vv}]\delta t.
\end{aligned} \tag{57}$$

The expressions for $\delta\mathfrak{B}'_u$, $\delta\mathfrak{B}'_v$, $\delta\mathfrak{C}'_u$, $\delta\mathfrak{C}'_v$ have been omitted because the relations

$$\mathfrak{B} + \mathfrak{C}' = 0, \qquad \mathfrak{B}' + \mathfrak{C}'' = 0$$

render them unnecessary. These relations may also be used for the purpose of simplifying some of the terms in (57).

If we assume again, for a moment, that $DD'' - 4D'^2$ is different from zero, we may think of D, D'', $\mathfrak{B}''$, $\mathfrak{C}$ and the eight derivatives of the first order of these quantities as being the twelve independent variables of the complete system which determines the invariants under consideration. The first four equations of the system, obtained by equating to zero the coefficients of κ_u, κ_v, λ_u, λ_v in the general expression for δf, where f is an arbitrary function of these variables, are known to be independent. In fact, the matrix of the 4×12 coefficients of these four equations contains the determinant of the fourth order, denoted above by Δ_4, which is known to be different from zero. The elements of this determinant are the coefficients of $\partial f/\partial D$, $\partial f/\partial D''$, $\partial f/\partial\mathfrak{B}''$, $\partial f/\partial\mathfrak{C}$ in the first four equations of the complete system. The matrix of the coefficients of $\partial f/\partial D_u$,

$\partial f/\partial D_v, \cdots \partial f/\partial \mathfrak{C}_v$ in the remaining six equations is

$$(58)\qquad \begin{vmatrix} -D & 0 & +D'' & 0 & +\mathfrak{B}'' & 0 & -2\mathfrak{C} & 0 \\ 0 & -D & -2D' & +D'' & +3\mathfrak{C}'' & +\mathfrak{B}'' & 0 & -2\mathfrak{C} \\ 0 & 0 & 0 & -2D' & 0 & +3\mathfrak{C}'' & 0 & 0 \\ -2D' & 0 & 0 & 0 & 0 & 0 & +3\mathfrak{B} & 0 \\ +D & -2D' & -D'' & 0 & -2\mathfrak{B}'' & 0 & +\mathfrak{C} & +3\mathfrak{B} \\ 0 & +D & 0 & -D'' & 0 & -2\mathfrak{B}'' & 0 & +\mathfrak{C} \end{vmatrix}$$

The independent variables u and v of system (6) may be so chosen that $D = D'' = 0$, $D' = 1$, in which case of course $DD'' - 4D'^2$ does not vanish. This is equivalent to the choice of the asymptotic curves of the integral surface as parameter lines. In that case the determinant of the sixth order Δ_6 obtained from the matrix (58) by suppressing the last two columns becomes equal to $\mathfrak{B}''^3$ except for a numerical factor. It does not vanish, therefore, unless the surface is ruled. In the matrix of all of the 10×12 coefficients of the ten equations of the complete system there occurs therefore a determinant of the tenth order which may be represented as follows:

$$(59)\qquad \Delta_{10} = \begin{vmatrix} \Delta_4 & \Delta' \\ 0 & \Delta_6 \end{vmatrix}$$

where Δ' represents a matrix of four rows and six columns, and where 0 represents a matrix of six rows and four columns, all of whose elements are equal to zero. Obviously

$$\Delta_{10} = \Delta_4 \cdot \Delta_6$$

and is therefore different from zero. Consequently the ten equations of the complete system are independent; and, as announced before, there exist two new relative invariants, distinct from θ, which involve no higher derivatives than the first of the quantities $\mathfrak{B}$, $\mathfrak{C}$, D.

In order to find these new invariants, we first determine from (56) a set of auxiliary quantities whose infinitesimal transformations depend only upon the first derivatives of κ and λ. This may be done as follows. In each of the eight equations (57) there occur only three of the second derivatives of κ and λ, and these may be eliminated by combining properly with the expressions for the infinitesimal transformations of P_u, P_v, R_u, R_v, S_u, S_v. For instance we have, from (53) and (57),

$$\delta\mathfrak{B}_u = [-2\mathfrak{B}_u\kappa_u + \mathfrak{C}_u\kappa_v + (2\mathfrak{C}_u'' - \mathfrak{B}_v)\lambda_u - \mathfrak{B}\kappa_{uu} + \mathfrak{C}\kappa_{uv} + 2\mathfrak{C}''\lambda_{uu}]\,\delta t,$$

$$\delta P_u = [-P_u\kappa_u - R_u\kappa_v - P_v\lambda_u - 2P_u\lambda_v - R\kappa_{uv} - 2P\lambda_{uv}]\,\delta t,$$

$$
\begin{aligned}
\delta R_u &= [-2R_u\kappa_u - 2S_u\kappa_v - (2P_u + R_v)\lambda_u - R_u\lambda_v - R\kappa_{uu} - 2S\kappa_{uv} - 2P\lambda_{uu} \\
&\qquad - R\lambda_{uv}]\,\delta t, \\
\delta S_u &= [-3S_u\kappa_u - (R_u + S_v)\lambda_u - 2S\kappa_{uu} - R\lambda_{uu}]\,\delta t, \\
\delta S_v &= [-2S_v\kappa_u - S_u\kappa_v - R_v\lambda_u - S_v\lambda_v - 2S\kappa_{uv} - R\lambda_{uv}]\,\delta t.
\end{aligned} \tag{60}
$$

We determine the ratios of five quantities a, b, c, d, e, in such a way that

$$a\delta\mathfrak{B}_v + b\delta P_u + c\delta R_u + d\delta S_u + e\delta S_v$$

shall be free from the second derivatives of κ and λ. Since κ_{vv} and λ_{vv} do not occur in any of the expressions (60), this condition gives us just four linear homogeneous equations for the four ratios. Their solution gives

$$a = \theta^2, \qquad b = \mathfrak{C}R, \qquad c = -(\mathfrak{B}R + 4\mathfrak{C}''S),$$

$$d = 2(\mathfrak{B}P + \mathfrak{C}''R), \qquad e = \mathfrak{B}R - 2\mathfrak{C}P + 4\mathfrak{C}''S.$$

If now

$$a\mathfrak{B}_u + bP_u + cR_u + dS_u + eS_v$$

be denoted by $\binom{\mathfrak{B}}{u}$, it is certain that the expression for $\delta\binom{\mathfrak{B}}{u}$ will contain no terms which contain the second derivatives of κ or λ as factors. In precisely similar fashion, seven other auxiliary quantities are introduced.

Consequently let

$$
\begin{aligned}
\binom{\mathfrak{B}}{u} &= \theta^2\mathfrak{B}_u + bP_u + cR_u + dS_u + eS_v, \\
\binom{\mathfrak{B}}{v} &= \theta^2\mathfrak{B}_v + bP_v + (c+e)R_v + dS_v - eP_u, \\
\binom{\mathfrak{B}''}{u} &= \theta^2\mathfrak{B}_u'' + b''P_u + c''R_u + d''S_u + e''S_v, \\
\binom{\mathfrak{B}''}{v} &= \theta^2\mathfrak{B}_v'' + b''P_v + (c''+e'')R_v + d''S_v - e''P_u, \\
\binom{\mathfrak{C}}{u} &= \theta^2\mathfrak{C}_u + mP_u + nR_u + pS_u + qS_v, \\
\binom{\mathfrak{C}}{v} &= \theta^2\mathfrak{C}_v + mP_v + (n+q)R_v + pS_v - qP_u, \\
\binom{\mathfrak{C}''}{u} &= \theta^2\mathfrak{C}_u'' + m''P_u + n''R_u + p''S_u + q''S_v, \\
\binom{\mathfrak{C}''}{v} &= \theta^2\mathfrak{C}_v'' + m''P_v + (n''+q'')R_v + p''S_v - q''P_u,
\end{aligned} \tag{61}
$$

where

$$b = \mathfrak{C}R, \qquad c = -(\mathfrak{B}R + 4\mathfrak{C}''S), \qquad d = 2(\mathfrak{B}P + \mathfrak{C}''R),$$

$$e = \mathfrak{B}R - 2\mathfrak{C}P + 4\mathfrak{C}''S, \qquad b'' = 3\mathfrak{C}''R + 4\mathfrak{B}''S, \qquad c'' = \mathfrak{B}''R,$$

$$
\begin{gathered}
d'' = -2\mathfrak{B}''P, \qquad e'' = -3(\mathfrak{B}''R + 2\mathfrak{C}''P), \\
m = -2\mathfrak{C}S, \qquad n = -2(\mathfrak{C}R + 3\mathfrak{B}S), \qquad p = 3\mathfrak{B}R + 4\mathfrak{C}P, \\
q = 3(\mathfrak{C}R + 2\mathfrak{B}S), \qquad m'' = 2(\mathfrak{B}R + \mathfrak{C}''S), \\
n'' = -2\mathfrak{B}''S, \qquad p'' = \mathfrak{B}''R, \qquad q'' = -\mathfrak{C}''R - 4\mathfrak{B}P + 2\mathfrak{B}''S.
\end{gathered} \tag{62}
$$

Then will

$$
\begin{aligned}
\delta\binom{\mathfrak{B}}{u} &= \left[-4\binom{\mathfrak{B}}{u}\kappa_u + \binom{\mathfrak{C}}{u}\kappa_v + \left\{2\binom{\mathfrak{C}''}{u} - \binom{\mathfrak{B}}{v}\right\}\lambda_u - 2\binom{\mathfrak{B}}{u}\lambda_v\right]\delta t, \\
\delta\binom{\mathfrak{B}''}{u} &= \left[-2\binom{\mathfrak{B}''}{u}\kappa_u + 3\binom{\mathfrak{C}''}{u}\kappa_v - \binom{\mathfrak{B}''}{v}\lambda_u - 4\binom{\mathfrak{B}''}{u}\lambda_v\right]\delta t, \\
\delta\binom{\mathfrak{C}}{u} &= \left[-5\binom{\mathfrak{C}}{u}\kappa_u + \left\{3\binom{\mathfrak{B}}{u} - \binom{\mathfrak{C}}{v}\right\}\lambda_u - \binom{\mathfrak{C}}{u}\lambda_v\right]\delta t, \\
\delta\binom{\mathfrak{C}''}{u} &= \left[-3\binom{\mathfrak{C}''}{u}\kappa_u + 2\binom{\mathfrak{B}}{u}\kappa_v + \left\{\binom{\mathfrak{B}''}{u} - \binom{\mathfrak{C}''}{v}\right\}\lambda_u - 3\binom{\mathfrak{C}''}{u}\lambda_v\right]\delta t, \\
\delta\binom{\mathfrak{B}}{v} &= \left[-3\binom{\mathfrak{B}}{v}\kappa_u + \left\{\binom{\mathfrak{C}}{v} - \binom{\mathfrak{B}}{u}\right\}\kappa_v + 2\binom{\mathfrak{C}''}{v}\lambda_u - 3\binom{\mathfrak{B}}{v}\lambda_v\right]\delta t, \\
\delta\binom{\mathfrak{B}''}{v} &= \left[-\binom{\mathfrak{B}''}{v}\kappa_u + \left\{3\binom{\mathfrak{C}''}{v} - \binom{\mathfrak{B}''}{u}\right\}\kappa_v - 5\binom{\mathfrak{B}''}{v}\lambda_v\right]\delta t, \\
\delta\binom{\mathfrak{C}}{v} &= \left[-4\binom{\mathfrak{C}}{v}\kappa_u - \binom{\mathfrak{C}}{u}\kappa_v + 3\binom{\mathfrak{B}}{v}\lambda_u - 2\binom{\mathfrak{C}}{v}\lambda_v\right]\delta t, \\
\delta\binom{\mathfrak{C}''}{v} &= \left[-2\binom{\mathfrak{C}''}{v}\kappa_u + \left\{2\binom{\mathfrak{B}}{v} - \binom{\mathfrak{C}''}{u}\right\}\kappa_v + \binom{\mathfrak{B}''}{v}\lambda_u - 4\binom{\mathfrak{C}''}{v}\lambda_v\right]\delta t.
\end{aligned} \tag{63}
$$

We denote by Φ and Ψ the following functions;

$$
\begin{aligned}
\Phi &= \left[\binom{\mathfrak{C}''}{u} + \binom{\mathfrak{B}}{v}\right]^2 - \left[\binom{\mathfrak{B}}{u} + \binom{\mathfrak{C}}{v}\right]\left[\binom{\mathfrak{B}''}{u} + \binom{\mathfrak{C}''}{v}\right], \\
\Psi &= -2\left[\binom{\mathfrak{C}''}{u} - \binom{\mathfrak{B}}{v}\right]^3 + \binom{\mathfrak{B}''}{v}\left[3\binom{\mathfrak{B}}{u} - \binom{\mathfrak{C}}{v}\right]^2 - \binom{\mathfrak{C}}{u}\left[\binom{\mathfrak{B}''}{u} - 3\binom{\mathfrak{C}''}{v}\right]^2 \\
&\quad + \left[\binom{\mathfrak{C}''}{u} - \binom{\mathfrak{B}}{v}\right]\left[3\binom{\mathfrak{B}}{u} - \binom{\mathfrak{C}}{v}\right]\left[\binom{\mathfrak{B}''}{u} - 3\binom{\mathfrak{C}''}{v}\right] \\
&\quad - 8\binom{\mathfrak{C}}{u}\binom{\mathfrak{B}''}{v}\left[\binom{\mathfrak{C}''}{u} - \binom{\mathfrak{B}}{v}\right].
\end{aligned} \tag{64}
$$

These quantities, Φ and Ψ, are invariants. In fact

$$
\delta\Phi = -6(\kappa_u + \lambda_v)\Phi\delta t, \qquad \delta\Psi = -9(\kappa_u + \lambda_v)\Psi\delta t, \tag{65}
$$

so that

$$\frac{\Phi}{\theta^6} \quad \textit{and} \quad \frac{\Psi}{\theta^9}$$

are absolute invariants. These are the two whose existence was demonstrated above.

The three invariants θ, Φ and Ψ are independent of the seminvariants A, A', A''. We shall now proceed to calculate invariants involving also these variables.

We again begin by forming from A, A', A'' a set of three quantities $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$, such that $\delta\mathfrak{A}$, $\delta\mathfrak{A}'$, $\delta\mathfrak{A}''$ shall contain only the first derivatives of κ and λ. Put

$$(66) \qquad E = \frac{\theta_{uu}}{\theta} - \frac{3}{2}\frac{\theta_u^2}{\theta^2}, \qquad F = \frac{\theta_{uv}}{\theta} - \frac{3}{2}\frac{\theta_u\theta_v}{\theta^2}, \qquad G = \frac{\theta_{vv}}{\theta} - \frac{3}{2}\frac{\theta_v^2}{\theta^2},$$

and let

$$\mathfrak{A} = A - \frac{1}{2}\left(B\frac{\theta_u}{\theta} + C\frac{\theta_v}{\theta}\right) + \tfrac{1}{2}TE - \tfrac{1}{2}DD'F + \tfrac{1}{4}D^2G,$$

$$(67) \qquad \mathfrak{A}' = A' - \frac{1}{2}\left(B'\frac{\theta_u}{\theta} + C'\frac{\theta_v}{\theta}\right) + \tfrac{1}{4}D'D''E - \tfrac{1}{2}DD''F + \tfrac{1}{4}DD'G,$$

$$\mathfrak{A}'' = A'' - \frac{1}{2}\left(B''\frac{\theta_u}{\theta} + C''\frac{\theta_v}{\theta}\right) + \tfrac{1}{4}D''^2E - \tfrac{1}{2}D'D''F + \tfrac{1}{2}TG,$$

where, as before

$$T = D'^2 - \tfrac{1}{2}DD''.$$

Then

$$\delta\mathfrak{A} = [-2\mathfrak{A}\kappa_u - 2\mathfrak{A}'\lambda_u]\,\delta t,$$

$$(68) \qquad \delta\mathfrak{A}' = [-\mathfrak{A}'\kappa_u - \mathfrak{A}\kappa_v - \mathfrak{A}''\lambda_u - \mathfrak{A}'\lambda_v]\,\delta t,$$

$$\delta\mathfrak{A}'' = [-2\mathfrak{A}'\kappa_v - 2\mathfrak{A}''\lambda_v]\,\delta t.$$

Notice that $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$ satisfy the same relation

$$(69) \qquad D''\mathfrak{A} - 2D'\mathfrak{A}' + D\mathfrak{A}'' = 0,$$

which is also satisfied by $\mathfrak{B}$, $\mathfrak{B}'$, $\mathfrak{B}''$ and $\mathfrak{C}$, $\mathfrak{C}'$, $\mathfrak{C}''$.

Equations (68) show that

$$(70) \qquad \Sigma = \mathfrak{A}'^2 - \mathfrak{A}\mathfrak{A}''$$

is an invariant for which

$$\delta\Sigma = -2(\kappa_u + \lambda_v)\Sigma\delta t.$$

Another invariant may be constructed as follows: Let

$$(71) \quad (\mathfrak{A}, \mathfrak{B}) = \mathfrak{A}\mathfrak{B}'' - 2\mathfrak{A}'\mathfrak{B}' + \mathfrak{A}''\mathfrak{B}, \qquad (\mathfrak{A}, \mathfrak{C}) = \mathfrak{A}\mathfrak{C}'' - 2\mathfrak{A}'\mathfrak{C}' + \mathfrak{A}''\mathfrak{C}.$$

Then

$$\delta(\mathfrak{A}, \mathfrak{B}) = [-(\mathfrak{A}, \mathfrak{B})\kappa_u + (\mathfrak{A}, \mathfrak{C})\kappa_v - 2(\mathfrak{A}, \mathfrak{B})\lambda_v]\,\delta t,$$

$$(72) \qquad \delta(\mathfrak{A}, \mathfrak{C}) = [-2(\mathfrak{A}, \mathfrak{C})\kappa_u + (\mathfrak{A}, \mathfrak{B})\lambda_u - (\mathfrak{A}, \mathfrak{C})\lambda_v]\,\delta t.$$

Consequently, if we put

$$\mathfrak{A}(\mathfrak{A},\mathfrak{B})+\mathfrak{A}'(\mathfrak{A},\mathfrak{C})=\mathfrak{F},\qquad \mathfrak{A}'(\mathfrak{A},\mathfrak{B})+\mathfrak{A}''(\mathfrak{A},\mathfrak{C})=\mathfrak{G},\tag{73}$$

we shall find

$$\delta\mathfrak{F}=[-3\mathfrak{F}\kappa_u-\mathfrak{G}\lambda_u-2\mathfrak{F}\lambda_v]\,\delta t,\quad \delta\mathfrak{G}=[-2\mathfrak{G}\kappa_u-\mathfrak{F}\kappa_v-3\mathfrak{G}\lambda_v]\,\delta t,\tag{74}$$

so that

$$\Omega=(\mathfrak{A},\mathfrak{B})\mathfrak{F}+(\mathfrak{A},\mathfrak{C})\mathfrak{G}=\mathfrak{A}(\mathfrak{A},\mathfrak{B})^2+2\mathfrak{A}'(\mathfrak{A},\mathfrak{B})(\mathfrak{A},\mathfrak{C})+\mathfrak{A}''(\mathfrak{A},\mathfrak{C})^2\tag{75}$$

is another invariant (of weight four), since

$$\delta\Omega=-4(\kappa_u+\lambda_v)\,\Omega\delta t.\tag{76}$$

From the invariants already obtained, others may be derived by differentiation processes which are very closely related to the differential parameters of the metrical theory of surfaces.

In fact, *let I and J be any two absolute invariants. Then*

$$\begin{aligned}
\Delta_1(I)&=\frac{1}{\theta}[D''I_u^2-2D'I_uI_v+DI_v^2],\\
\Delta_1(J)&=\frac{1}{\theta}[D''J_u^2-2D'J_uJ_v+DJ_v^2],\\
\Delta(I,J)&=\frac{1}{\theta}[D''I_uJ_u-D'(I_uJ_v+I_vJ_u)+DI_vJ_v],\\
\Delta_2(I)&=\frac{1}{\theta}\left[\frac{\partial}{\partial u}(D''I_u-D'I_v)+\frac{\partial}{\partial v}(DI_v-D'I_u)\right],\\
\Delta_2(J)&=\frac{1}{\theta}\left[\frac{\partial}{\partial u}(D''J_u-D'J_v)+\frac{\partial}{\partial v}(DJ_v-D'J_u)\right]
\end{aligned}\tag{77}$$

are again absolute invariants, as is also the Jacobian

$$\Gamma(I,J)=\frac{1}{\theta}(I_uJ_v-I_vJ_u).\tag{78}$$

It is easy to verify that $\mathfrak{A}/\theta$, $\mathfrak{A}'/\theta$, $\mathfrak{A}''/\theta$ are cogredient with D, D', D''. Consequently *the quantities*

$$\begin{aligned}
\Lambda_1(I)&=\frac{1}{\theta^2}[\mathfrak{A}''I_u^2-2\mathfrak{A}'I_uI_v+\mathfrak{A}I_v^2],\\
\Lambda_1(J)&=\frac{1}{\theta^2}[\mathfrak{A}''J_u^2-2\mathfrak{A}'J_uJ_v+\mathfrak{A}J_v^2],\\
\Lambda(I,J)&=\frac{1}{\theta^2}[\mathfrak{A}''I_uJ_u-\mathfrak{A}'(I_uJ_v+I_vJ_u)+\mathfrak{A}I_vJ_v],
\end{aligned}\tag{79}$$

$$\Lambda_2(I)=\frac{1}{\theta}\left[\frac{\partial}{\partial u}\left(\frac{\mathfrak{A}''I_u-\mathfrak{A}'I_v}{\theta}\right)+\frac{\partial}{\partial v}\left(\frac{\mathfrak{A}I_v-\mathfrak{A}'I_u}{\theta}\right)\right],$$

$$\Lambda_2(J)=\frac{1}{\theta}\left[\frac{\partial}{\partial u}\left(\frac{\mathfrak{A}''J_u-\mathfrak{A}'J_v}{\theta}\right)+\frac{\partial}{\partial v}\left(\frac{\mathfrak{A}J_v-\mathfrak{A}'J_u}{\theta}\right)\right]$$

are also absolute invariants.

Each of these differentiation processes may be repeated, or combined with each of the others so as to produce a large number of invariants. But not all of the invariants obtained in this way are functionally independent. In order to obtain the relations between them, it is convenient to write down the simplified form which they assume if system (6) is assumed to be in its canonical form, for which

$$D=0 \qquad D'=1 \qquad D''=0, \tag{80}$$

so that

$$\mathfrak{A}'=0, \qquad \mathfrak{B}=\mathfrak{B}'=0, \qquad \mathfrak{C}'=\mathfrak{C}''=0, \tag{81}$$

and consequently further

$$A'=0, \qquad B=B'=0, \qquad C'=C''=0,$$

$$\mathfrak{B}''=B'', \qquad \mathfrak{C}=C, \qquad \theta=-B''C,$$

$$\mathfrak{A}=A-\tfrac{1}{2}C\frac{\theta_v}{\theta}+\frac{1}{2}\left(\frac{\theta_{uu}}{\theta}-\frac{3}{2}\frac{\theta_u^2}{\theta}\right), \tag{82}$$

$$\mathfrak{A}''=A''-\tfrac{1}{2}B''\frac{\theta_u}{\theta}+\frac{1}{2}\left(\frac{\theta_{vv}}{\theta}-\frac{3}{2}\frac{\theta_v^2}{\theta^2}\right).$$

The above derived differential invariants of the first order become

$$\Delta_1(I)=-\frac{2}{\theta}I_uI_v, \quad \Delta_1(J)=-\frac{2}{\theta}J_uJ_v, \quad \Delta(I,J)=-\frac{1}{\theta}(I_uJ_v+I_vJ_u),$$

$$\Lambda_1(I)=\frac{1}{\theta^2}(\mathfrak{A}''I_u^2+\mathfrak{A}I_v^2), \qquad \Lambda_1(J)=\frac{1}{\theta^2}(\mathfrak{A}''J_u^2+\mathfrak{A}J_v^2), \tag{83}$$

$$\Lambda(I,J)=\frac{1}{\theta^2}(\mathfrak{A}''I_uJ_u+\mathfrak{A}I_vJ_v), \qquad \Gamma(I,J)=\frac{1}{\theta}(I_uJ_v-I_vJ_u),$$

while those of the second order reduce to

$$\Delta_2(I)=-\frac{2}{\theta}I_{uv}, \qquad \Delta_2(J)=-\frac{2}{\theta}J_{uv},$$

$$\Lambda_2(I)=\frac{1}{\theta}\left[\frac{\partial}{\partial u}\left(\frac{\mathfrak{A}''I_u}{\theta}\right)+\frac{\partial}{\partial v}\left(\frac{\mathfrak{A}I_v}{\theta}\right)\right], \tag{84}$$

$$\Lambda_2(J)=\frac{1}{\theta}\left[\frac{\partial}{\partial u}\left(\frac{\mathfrak{A}''J_u}{\theta}\right)+\frac{\partial}{\partial v}\left(\frac{\mathfrak{A}J_v}{\theta}\right)\right].$$

From (83) *we see at once that*

$$\Delta(I, J)^2 - \Gamma(I, J)^2 - \Delta_1(I)\Delta_1(J) = 0. \tag{85}$$

Further we find

$$\begin{aligned} -\Delta(I, J)^2\Gamma(I, J)^2\mathfrak{A} &= -J_u^2\Lambda_1(I) + I_u^2\Lambda_1(J), \\ -\Delta(I, J)^2\Gamma(I, J)^2\mathfrak{A}'' &= +J_v^2\Lambda_1(I) - I_v^2\Lambda_1(J), \end{aligned} \tag{86}$$

which gives, upon substitution into the expression for $\Lambda(I, J)$, *the further relation*

$$2\Delta(I, J)\Lambda(I, J) - \Delta_1(I)\Lambda_1(J) - \Delta_1(J)\Lambda_1(I) = 0. \tag{87}$$

We shall not, in the present paper, investigate the relations between the operators of the second order and those obtained by repetition of the operators of the first order; nor shall we inquire into the question whether all invariants of the given system of differential equations can be obtained in this way. Let us confine our attention to those invariants which contain no derivatives of higher than the second order of the quantities $\mathfrak{B}$, $\mathfrak{B}'$, $\mathfrak{B}''$, $\mathfrak{C}$, $\mathfrak{C}'$, $\mathfrak{C}''$, D, D', D'', and which involve the quantities $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$ themselves but none of their derivatives. We now proceed to show that we are already in possession of a functionally complete set of invariants of this kind.

The invariants contemplated depend upon 26 independent arguments, viz.: four independent variables among the nine quantities $\mathfrak{B}, \cdots, D''$, their eight first, their twelve second derivatives, and two independent variables among the three $\mathfrak{A}$'s.

The infinitesimal transformations of these 26 variables involve terms containing the first, second, and third derivatives of κ and λ as factors. The absolute invariants of the class considered must therefore satisfy a complete system of 18 equations and 26 independent variables. If these equations are all independent, *there will be* $26 - 18 = 8$ *such invariants. They may be chosen as follows:*

$$\frac{\Phi}{\theta^6}, \quad \frac{\Psi}{\theta^9},$$

$$\Delta_1\left(\frac{\Phi}{\theta^6}\right), \quad \Delta_1\left(\frac{\Psi}{\theta^9}\right), \quad \Delta\left(\frac{\Phi}{\theta^6}, \frac{\Psi}{\theta^9}\right), \tag{88}$$

$$\Lambda_1\left(\frac{\Phi}{\theta^6}\right), \quad \Lambda_1\left(\frac{\Psi}{\theta^9}\right), \quad \frac{\Omega}{\theta^4}.$$

It remains to prove that the eighteen equations just mentioned are independent, and that the same is true of the eight invariants (88).

In the matrix formed by the coefficients of the first ten equations of our complete system, there occurs the determinant Δ_{10} of equation (59), which is known to be different from zero. The last eight equations, obtained by equating to

zero the coefficients of the third derivatives of κ and λ in the general expression for δf, contain only the twelve independent second derivatives of the $\mathfrak{B}$'s, $\mathfrak{C}$'s, and D's. If we assume again for the moment that $DD'' - 4D'^2$ does not vanish, we may think specifically of D, D'', $\mathfrak{B}''$ and $\mathfrak{C}$ together with their derivatives as the independent variables. The matrix of the 12×8 coefficients of the last eight equations will then be as follows:

$$\begin{array}{cccccccccccc}
-D & 0 & 0 & +D'' & 0 & 0 & +\mathfrak{B}'' & 0 & 0 & -2\mathfrak{C} & 0 & 0 \\
0 & -D & 0 & -2D' & +D'' & 0 & +3\mathfrak{C}'' & +\mathfrak{B}'' & 0 & 0 & -2\mathfrak{C} & 0 \\
0 & 0 & -D & 0 & -2D' & +D'' & 0 & +3\mathfrak{C}'' & +\mathfrak{B}'' & 0 & 0 & -2\mathfrak{C} \\
0 & 0 & 0 & 0 & 0 & -2D' & 0 & 0 & +3\mathfrak{C}'' & 0 & 0 & 0 \\
-2D' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +3\mathfrak{B} & 0 & 0 \\
+D & -2D' & 0 & -D'' & 0 & 0 & -2\mathfrak{B}'' & 0 & 0 & +\mathfrak{C} & +3\mathfrak{B} & 0 \\
0 & +D & -2D' & 0 & -D'' & 0 & 0 & -2\mathfrak{B}'' & 0 & 0 & +\mathfrak{C} & +3\mathfrak{B} \\
0 & 0 & +D & 0 & 0 & -D'' & 0 & 0 & -2\mathfrak{B}'' & 0 & 0 & +\mathfrak{C}
\end{array}$$

If the system (6) is in its canonical form, so that equations (80), (81) and (82) are satisfied, the determinant of the eighth order, obtained from the above matrix by omitting the eighth, tenth, eleventh, and twelfth columns, becomes equal to B''^2 multiplied by a power of 2. This determinant Δ_8 is therefore not identically equal to zero.

If we now consider the whole system of eighteen equations there will be in the matrix of its coefficients a determinant of order eighteen of the form

$$\Delta_{18} = \begin{vmatrix} \Delta_{10} & \Delta'' \\ 0 & \Delta_8 \end{vmatrix} = \Delta_{10} \cdot \Delta_8, \tag{89}$$

where Δ'' denotes a rectangular matrix of ten rows and eight columns, while 0 denotes a rectangular matrix of eight rows and ten columns all of whose elements are equal to zero. This determinant Δ_{18} does not vanish identically; consequently the eighteen equations of the complete system are independent.

The eight invariants (88) are obviously members of the class here considered. It only remains to show that they are independent.

In order to do this we again make use of the canonical form. According to (80), (81), (82), the canonical forms of the eight invariants are as follows:

$$\begin{aligned}
\frac{\Phi}{\theta^6} &= \frac{(2CB''_u + B''C_u)(2B''C_v + CB''_v)}{B''^3 C^3}, \\
\frac{\Psi}{\theta^9} &= -\frac{2CB''_u + B''C_u}{B''^4 C^5} + \frac{2B''C_v + CB''_v}{B''^5 C^4},
\end{aligned} \tag{90}$$

$$\Delta_1\left(\frac{\Phi}{\theta^6}\right)=\frac{2}{B''C}\frac{\partial}{\partial u}\left(\frac{\Phi}{\theta^6}\right)\frac{\partial}{\partial v}\left(\frac{\Phi}{\theta^6}\right),$$

$$\Delta_1\left(\frac{\Psi}{\theta^9}\right)=\frac{2}{B''C}\frac{\partial}{\partial u}\left(\frac{\Psi}{\theta^9}\right)\frac{\partial}{\partial v}\left(\frac{\Psi}{\theta^9}\right),$$

$$\Delta\left(\frac{\Phi}{\theta^6},\frac{\Psi}{\theta^9}\right)=\frac{1}{B''C}\left[\frac{\partial}{\partial u}\left(\frac{\Phi}{\theta^6}\right)\frac{\partial}{\partial v}\left(\frac{\Psi}{\theta^9}\right)+\frac{\partial}{\partial v}\left(\frac{\Phi}{\theta^6}\right)\frac{\partial}{\partial u}\left(\frac{\Psi}{\theta^9}\right)\right], \tag{90}$$

$$\Lambda_1\left(\frac{\Phi}{\theta^6}\right)=\frac{1}{(B''C)^2}\left[\mathfrak{A}''\left\{\frac{\partial}{\partial u}\left(\frac{\Phi}{\theta^6}\right)\right\}^2+\mathfrak{A}\left\{\frac{\partial}{\partial v}\left(\frac{\Phi}{\theta^6}\right)\right\}^2\right],$$

$$\Lambda_1\left(\frac{\Psi}{\theta^9}\right)=\frac{1}{(B''C)^2}\left[\mathfrak{A}''\left\{\frac{\partial}{\partial u}\left(\frac{\Psi}{\theta^9}\right)\right\}^2+\mathfrak{A}\left\{\frac{\partial}{\partial v}\left(\frac{\Psi}{\theta^9}\right)\right\}^2\right],$$

$$\frac{\Omega}{\theta^4}=\frac{\mathfrak{A}^3B''^2+\mathfrak{A}''^3C^2}{(B''C)^4}.$$

In the canonical form, our system of differential equations (6) becomes

$$\begin{aligned} y_{uu}&=Ay+Cy_v,\\ y_{vv}&=A''y+B''y_u, \end{aligned} \tag{91}$$

If we compare this with the system as written in the First and Second Memoirs, viz.:*

$$\begin{aligned} y_{uu}+2by_v+fy&=0,\\ y_{vv}+2a'y_u+gy&=0, \end{aligned} \tag{92}$$

we see that the quantities A, A'', B'', C correspond to $-f$, $-g$, $-2a'$, $-2b$ respectively. Let us remember further that in the Second Memoir† we introduced two absolute invariants, I and J, defined by the equations:

$$I=\frac{a'b_u+2a'_ub}{4a'b\sqrt[4]{a'b^2}},\qquad J=\frac{a'_vb+2a'b_v}{4a'b\sqrt[4]{a'^2b}}. \tag{93}$$

We thus find

$$\frac{\Phi}{4\theta^6}=IJ,\qquad \frac{\Psi}{8\theta^9}=J^3-I^3. \tag{94}$$

Thus the first two invariants of (90) are functions of I and J alone. The next three are independent combinations of the first derivatives of I and J, containing besides the single combination $B''C$ of B'' and C. The following two invariants

$$\Lambda_1\left(\frac{\Phi}{\theta^6}\right)\qquad\text{and}\qquad\Lambda_1\left(\frac{\Psi}{\theta^9}\right)$$

* These Transactions, vol. 8 (1907), p. 246.

† These Transactions, vol. 9 (1908), p. 103.

depend upon the same quantities, but contain the two further variables $\mathfrak{A}$ and $\mathfrak{A}''$ in two independent combinations. Finally Ω/θ^4 is independent of all the rest because it contains B'' and C in a combination independent of $B''C$. This completes the proof that the eight functions (89) form a functionally complete system of invariants of the order considered.

We have found, moreover, the relations (94) between the rational invariants θ, Φ, Ψ and the irrational invariants I and J which present themselves in the canonical development

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots$$

of a surface in the vicinity of an ordinary point.

The relation of the other invariants obtained in this paper to those considered in the previous memoirs, the question in regard to the completeness of our system of invariants and differential parameters, the determination of the covariants and the introduction of the adjoined system in its general form must be left for a later occasion.

THE UNIVERSITY OF ILLINOIS,
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NATURAL FAMILIES OF TRAJECTORIES: CONSERVATIVE FIELDS OF FORCE*

BY

EDWARD KASNER

INTRODUCTION.

The families of curves considered in this paper may be defined as the extremals connected with variation problems of the form

$$\int F ds = \text{minimum}, \tag{1}$$

where F is any point function and ds is the element of arc in the space considered.† In the plane such a family contains ∞^2 curves; in ordinary space, the number is ∞^4; and in the general case of n dimensions the number is $\infty^{2(n-1)}$, one passing through every point in every direction. The results obtained hold for the general case, though for simplicity they will be written out in the following only for three dimensions.

Any system of curves connected with an integral (1) we shall refer to as a *natural* system or family. The name seems justified in view of the many important geometrical and physical investigations which lead to variation problems of type (1). The following six are perhaps the most noteworthy.

1) *Trajectories* in a conservative field of force. Let a unit particle move in a field due to the work function (negative potential) $W(x, y, z)$. The totality of motions, for all initial conditions, given by the equations

$$\ddot{x} = W_x, \qquad \ddot{y} = W_y, \qquad \ddot{z} = W_z,$$

leads to ∞^5 trajectories. These may be grouped according to the values of h, the constant of energy defined by the energy equation

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2(W + h).$$

For each value of h we obtain a family of ∞^4 trajectories. Such a family is a

* Presented to the Society April 28, 1905 and October 30, 1906, in two papers entitled *Velocity systems in the dynamics of a particle* and *The motion of a particle under conservative forces.*

† The space is supposed to be euclidean of any dimensionality. The results admit of direct extension to all spaces of constant curvature. Geodesic circles then play the rôle of ordinary circles.

natural family, since by the principle of least action we have

$$\int \sqrt{W+h}\, ds = \text{minimum}.$$

The complete system* of ∞^5 trajectories in a conservative field is thus composed of ∞^1 natural families.

2) *Brachistochrones.* In the case of conservative forces the brachistochrone problem leads to the integral

$$\int dt = \int \frac{ds}{\sqrt{W+h}}.$$

The complete system of brachistochrones is thus made up of ∞^1 natural families.

3) *Catenaries.* When a homogeneous flexible inextensible string is acted on by conservative forces, the forms of equilibrium (catenaries in the general sense) are determined by rendering

$$\int (W+h)ds$$

a minimum. Hence here also we have ∞^1 natural families, one for each value of h.†

4) *Optics.* Consider an isotropic medium in which the index of refraction ν varies from point to point. The paths of light in such a medium, according to FOURIER's principle, are determined by minimizing the integral

$$\int \nu ds,$$

and hence form a single natural family.

5) *Geodesics.* On a surface whose squared element of length (first fundamental form) is

$$\lambda(x, y)(dx^2 + dy^2),$$

the determination of the geodesics leads to the minimizing of

$$\int \sqrt{\lambda}\, ds;$$

hence the representing curves in the x, y plane form a natural family. Thus if any surface is represented conformally on a plane the geodesics are pictured by a natural family of curves in that plane. The extension to more variables is obvious: any natural family in any space may be obtained by conformal representation from the geodesics of some variety in higher space.

*Such complete systems, for both conservative and non-conservative forces, were treated in the author's earlier papers in these Transactions, vol. 7 (1906), pp. 401–424, and vol. 8 (1907), pp. 135–158.

† The total systems of ∞^5 catenaries and ∞^5 brachistochrones have geometric properties distinct from each other and from those of the ∞^5 trajectories. This subject will be treated in another paper.

6) *Contact transformations.* LIE * has indicated the importance of that category of infinitesimal contact transformation, whose characteristic function is, in the case of the plane, of the particular form

$$\Omega(x, y)\sqrt{1 + y'^2}.$$

These are characterized by the fact that the elements at each point are converted into the elements of a circle about that point as center. The path curves of such a transformation form a natural family.

The principal object of this paper is to obtain a complete geometric characterization of natural families of curves. It is shown that two properties, referred to as A and B, are necessary and sufficient. The first states that if the osculating circles of those curves of the family which pass through a given point p are constructed at that point, they will have a second point P in common; and thus (in the case of three dimensions) form a bundle. It is shown as a consequence of A that three (or n in the case of n dimensions) of the circles in such a bundle will have four-point contact with the corresponding curves. Property B states that these hyperosculating circles will be mutually orthogonal.

The next result is a *reciprocity principle* for natural systems. With each natural system S there is associated a second natural system S' as follows: the bundle of osculating circles constructed at a point p for the S curves is at the same time the bundle of osculating circles at the second point P of the bundle for the S' curves. The relation is of course mutual.

The correspondence of p and P leads to a point transformation whose character may be described in terms of a known category of space deformations referred to as of the DARBOUX type. The result is an equivalent for property B.

The class of systems having property A is of course far more extensive than the class of natural systems. All such systems are here termed *velocity systems.* They arise in fact in the study of certain curves (*velocity curves*) which are definitely related to the motion of a particle in an arbitrary field of force. For a given value of the speed, $v = v_0$, there are ∞^4 of these curves; they are defined by the fact that if a particle is projected in the direction of one of them with speed v_0 the trajectory described will osculate the velocity curve.

The last part of the paper is devoted to the two-dimensional theory. Here the velocity type includes not only the natural type but also the isogonal type (formed by all the ∞^2 isogonals of a given set of ∞^1 curves). The comparison

* *Die infinitesimalen Berührungstransformationen der Mechanik*, Leipziger Berichte (1889), pp. 145–153, and LIE-SCHEFFERS, *Geometrie der Berührungstransformationen*, p. 102. A more complete treatment is given by VESSIOT, Bulletin de la Societé Mathematique de France, vol. 34 (1906), pp. 230–269. The present writer has shown that the alternant of any two of these contact transformations is a point transformation. See Bulletin of the American Mathematical Society, vol. 14 (1908), p. 356.

of these two types leads to simple results. A transformation of curvature elements is found which interchanges the types. SCHEFFER'S theory of isogonal trajectories * is supplemented with respect to an essential feature.

§ 1. OSCULATING CIRCLES — PROPERTY A.

The extremals connected with an integral of form (1), which in the case of three dimensions we write

$$\int F(x, y, z)\sqrt{1 + y'^2 + z'^2}\,dx, \tag{1'}$$

are given by the EULER equations †

$$\begin{aligned} y'' &= (L_y - y'L_x)(1 + y'^2 + z'^2), \\ z'' &= (L_z - z'L_x)(1 + y'^2 + z'^2), \end{aligned} \tag{2}$$

where

$$L = \log F. \tag{2'}$$

Thus any natural family of curves is represented by differential equations of the form (2), *where* L *is an arbitrary point function.*

Of the ∞^4 curves in such a family, ∞^2 pass through any given point, one in each direction. Our first result is

THEOREM 1. *The* ∞^4 *curves of any natural family have this property: the circles which osculate at any point* p *of space the various curves passing through that point have a second point* P *in common, and thus form a bundle.*

This property we shall refer to as *property* A. In the discussion it will be convenient to decompose it into these two statements, which also relate to the ∞^2 curves through a given point:

(A_1) The osculating planes constructed at the common point form a pencil.

(A_2) The centers of curvature lie in a plane perpendicular to the axis of the pencil of osculating planes.

A synthetic proof of the theorem stated is easily obtained from the dynamical interpretation 1) of the introduction. Property A_1 results from the fact that the osculating plane of a trajectory always passes through the force vector. Property A_2 is proved by noting that the trajectories through a given point which correspond to the same value of the total energy h are all described with the same initial velocity v_0. The radius of curvature at the initial point is

$$r = v_0^2/N,$$

where N denotes the component of the force along the principal normal. Since N is the orthogonal projection of a fixed vector, the locus of its terminal point

* Leipziger Berichte, 1898 and 1900; Mathematische Annalen, vol. 60 (1905).

† Throughout the paper primes refer to total derivatives with respect to x; and literal subscripts to partial derivatives.

will be a sphere through the initial point. The conclusion then follows from the fact that r varies inversely as N.

The following analytical discussion has the advantage of answering the converse question which naturally arises: Are there other systems with property A?

The differential equations of any system of ∞^4 space curves, one determined by each lineal element of space, may be assumed in the form

$$y'' = f(x, y, z, y', z'), \qquad z'' = g(x, y, z, y', z'). \tag{3}$$

Property A_1 requires that at each point there shall be a certain direction through which all the osculating planes at that point must pass. Let the direction in question be given by the ratios of three arbitrary point functions

$$\phi(x, y, z), \qquad \psi(x, y, z), \qquad \chi(x, y, z); \tag{4}$$

then the requisite condition is

$$\frac{z''}{y''} = \frac{\chi - z'\phi}{\psi - y'\phi}. \tag{5}$$

Property A_2 requires that the center of curvature shall lie in a plane perpendicular to the direction (4); hence

$$\phi X + \psi Y + \chi Z = 1, \tag{6}$$

where X, Y, Z denote the coördinates of the center relative to axes with the common point as origin. Using the general formulas for the center of curvature, and combining with (5), we find

THEOREM 2. *The differential equations of any system of curves possessing property A are of the form*

$$\begin{aligned} y'' &= (\psi - y'\phi)(1 + y'^2 + z'^2), \\ z' &= (\chi - z'\phi)(1 + y'^2 + z'^2), \end{aligned} \tag{7}$$

where ϕ, ψ, χ are arbitrary functions of x, y, z. The converse is valid also.

The equations (2) are seen to be included in this form, hence the result certainly holds for our natural systems, as stated in theorem 1.

§ 2. HYPEROSCULATIONS—PROPERTY B.

The circles of curvature at a given point, for any system of the form (7), constitute a bundle. We now inquire whether any of these circles correspond to four-point, instead of three-point, contact.

If a twisted curve is to have an hyperosculating circle of curvature at a given point two conditions must be satisfied, namely,

$$\begin{vmatrix} 1 & y' & z' \\ 0 & y'' & z'' \\ 0 & y''' & z''' \end{vmatrix} = 0, \tag{8}$$

$$\frac{dr}{ds} = 0. \tag{9}$$

The first of these states that the osculating plane has four-point contact with the curve; the second, in which r denotes the radius of curvature, is the condition for the existence of an osculating helix, i. e., one with four-point contact. When both conditions hold the helix is simply the circle of curvature which then has hyper-contact.

Applying these conditions to the curves defined by (7), we find, from (8),

$$(\psi - y'\phi)\chi' - (\chi - z'\phi)\psi' + (y'\chi - z'\psi)\phi' = 0; \tag{10}$$

and,* from (9),

$$(1+y'^2+z'^2)\Sigma\phi\phi' - (\phi+y'\psi+z'\chi)\left\{\begin{matrix}(1+y'^2+z'^2)\Sigma\phi^2+\phi'+y'\psi'+z'\chi' \\ -(\phi+y'\psi+z'\chi)^2\end{matrix}\right\} = 0, \tag{11}$$

where the indicated summations extend over ϕ, ψ, χ and where ϕ', for example, denotes $\phi_x + y'\phi_y + z'\phi_z$.

Since we wish to discuss the ∞^2 curves through a given point, we may simplify our equations considerably by taking the axis of abscissas in the special direction (4). Then, at the selected point, ψ and χ vanish, and the above equations reduce to

$$y'\chi' - z'\psi' = 0, \tag{10'}$$

$$(y'^2 + z'^2)(\phi' - \phi^2) - (y'\psi' + z'\chi') = 0. \tag{11'}$$

Neglecting the trivial solutions for which $y'^2 + z'^2$ vanishes, we may reduce this pair of simultaneous equations to the form

$$\frac{\psi_x + y'\psi_y + z'\psi_z}{y'} = \frac{\chi_x + y'\chi_y + z'\chi_z}{z'} = \phi_x + y'\phi_y + z'\phi_z - \phi^2. \tag{12}$$

This set of equations for the determination of y', z' is of a familiar type, namely, that arising in the determination of the fixed points of a collineation, and is easily shown to admit three solutions.† Hence

THEOREM 3. *The curves defined by equations of the form* (7) *are such that through each point there pass three with hyperosculating circles at that point.*

Since the form (7) is characterized by property A, it follows that the existence of three hyperosculating circles in each bundle is a consequence of property A.

* The radius of curvature for the curves (7) is determined by

$$\frac{1}{r^2} = \phi^2 + \psi^2 + \chi^2 - \frac{(\phi + y'\psi + z'\chi)^2}{1 + y'^2 + z'^2}.$$

† Of course in special cases some of these may coincide, or the number of solutions may become infinite. The theorem stated is true "in general" in so far as it omits these cases which are definitely assignable.

We state two further properties, found by considering the conditions (10′) and (11′) separately.

The tangents to those curves of a system (7) *which pass through a given point and there have an hyperosculating plane form a quadric cone.* This cone passes through the special direction (4).

The tangents to those curves which have an osculating helix at the given point form a cubic cone. This cone passes through the special direction (4) and through the minimal directions in the plane normal to that direction.

These properties hold for natural families since they hold for all systems with property A. By comparing (7) with (2), we see that the functions ϕ, ψ, χ in the case of a natural family are

$$\phi = L_x, \qquad \psi = L_y, \qquad \chi = L_z; \tag{13}$$

and hence are connected by the relations

$$\psi_z - \chi_y = 0, \qquad \chi_x - \phi_z = 0, \qquad \phi_y - \psi_x = 0. \tag{14}$$

We now inquire what is the effect of these relations on the directions of the hyperosculating circles. Introducing, for symmetry,

$$X : Y : Z = 1 : y' : z', \tag{15}$$

we may write our equations (12) in the homogeneous form

$$\begin{aligned} &\chi_y Y^2 - \psi_z Z^2 + (\chi_x - \psi_y) YZ &&+ \chi_z XY - \psi_z XZ = 0, \\ &-\chi_x X^2 &&+ \phi_z Z^2 + \phi_y YZ - \chi_y XY + (\phi_x - \phi^2 - \chi_z) XZ = 0, \\ &\psi_x X^2 - \phi_y Y^2 &&- \phi_z YZ - (\phi_x - \phi^2 - \psi_y) XY + \psi_z XZ = 0. \end{aligned} \tag{16}$$

In virtue of (14), each of the quadric cones (16) is seen to be of the rectangular type.* Hence the three generators common to the cones must be mutually orthogonal. This gives

THEOREM 4. *In the case of any natural family the three hyperosculating circles which exist in any bundle are mutually orthogonal.*

We refer to this property as *property* B.

The relations (14) are seen to be necessary as well as sufficient for the orthogonality in question. Hence property B is the equivalent of (14), and serves to single out the natural families from the more general class defined by equations of form (7). The latter form was characterized by property A; hence we have our fundamental

THEOREM 5. *A system of* ∞^4 *curves, one for each direction at each point of space, will constitute a natural family when, and only when, it possesses prop-*

* The condition for such a cone is that the sum of the coefficients of X^2, Y^2, and Z^2 shall vanish.

erties A and B; that is, the osculating circles at any given point must form a bundle, and the three hyperosculating circles contained in such a bundle must be mutually orthogonal.

§ 3. Reciprocal Systems in General.

We have seen that the most general system of curves with property A is represented by equations of the form

$$(7) \qquad y'' = (\psi - y'\phi)(1 + y'^2 + z'^2), \qquad z'' = (\chi - z'\phi)(1 + y'^2 + z'^2),$$

where ϕ, ψ, χ are any functions of x, y, z. Such a system S induces a definite point correspondence: the osculating circles at a given point p pass through a second common point P. From the formulas given it is easily found that the coördinates of P are

$$(17) \quad X = x + \frac{2\phi}{\phi^2 + \psi^2 + \chi^2}, \quad Y = y + \frac{2\psi}{\phi^2 + \psi^2 + \chi^2}, \quad Z = z + \frac{2\chi}{\phi^2 + \psi^2 + \chi^2},$$

where x, y, z are of course the coördinates of p. This point transformation, from p to P, we denote by T.

It is easily seen that T is entirely general. To an arbitrary point transformation (which may even degenerate but must not be merely identity)

$$(18) \qquad X = X(x, y, z), \qquad Y = Y(x, y, z), \qquad Z = Z(x, y, z)$$

corresponds in fact a definite system (7), namely, that system for which

$$(19) \qquad \phi, \psi, \chi = \frac{2(X - x),\ 2(Y - y),\ 2(Z - z)}{\Sigma(X - x)^2}.$$

We assume in the remainder of the paper that the jacobian J of (17) does not vanish identically, so that we may consider the inverse transformation T^{-1}. To this corresponds a definite system S', which may be termed the *reciprocal* of the given system S.

Theorem 6. *For any system S with property A, that is any system* (7), *there is a definite reciprocal system S' of that type (provided, of course, J does not vanish). The osculating circles of those curves of the system S which pass through a point p are at the corresponding point P the osculating circles of the curves of the system S' passing through P.*

Consider the bundle of circles determined by two corresponding points p and P. These circles osculate the S curves at p and the S' curves at P. We know that three of these circles have hyper-contact with S curves. There must also be three which have hyper-contact with S' curves. We now prove that these wo triples of circles actually coincide.

We take the point p as origin and select our axes and unit of length so that, at that point, $\phi = 2$, $\psi = 0$, $\chi = 0$. Then from (17) the point P is $X = 1$, $Y = 0$, $Z = 0$. The directions of the hyperosculating circles at p are given by equations (12), which now take the form

$$(20) \qquad \frac{\chi_x + y'\chi_y + z'\chi_z}{z'} = \frac{\psi_x + y'\psi_y + z'\psi_z}{y'} = \phi_x + y'\phi_y + z'\phi_z - 4.$$

From (19), we find the values, at the origin, of the partial derivatives here involved, namely,

$$(21) \qquad \begin{aligned} &\phi_x = 2 - 2X_x, && \phi_y = -2X_y, && \phi_z = -2X_z, \\ &\psi_x = 2Y_x, && \psi_y = -2 + 2Y_y, && \psi_z = 2Y_z, \\ &\chi_x = 2Z_x, && \chi_y = 2Z_y, && \chi_z = -2 + 2Z_z. \end{aligned}$$

Hence (20) may be written

$$(22) \qquad \frac{Z_x + y'Z_y + z'(Z_z - 1)}{z'} = \frac{Y_x + y'(Y_y - 1) + z'Y_z}{y'} = -\{(X_x - 1) + y'X_y + z'X_z\} - 2,$$

which may be reduced to the form

$$(23) \qquad -(a_1 + a_2y' + a_3z') = \frac{b_1 + b_2y' + b_3z'}{y'} = \frac{c_1 + c_2y' + c_3z'}{z'},$$

where the elements of the matrix

$$(24) \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

stand for the elements of the jacobian

$$(25) \qquad J \equiv \begin{vmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{vmatrix}.$$

In the reciprocal system S' we must consider x, y, z as functions of X, Y, Z. Hence (25) is replaced by

$$(26) \qquad \begin{vmatrix} x_X & x_Y & x_Z \\ y_X & y_Y & y_Z \\ z_X & z_Y & z_Z \end{vmatrix} \equiv \frac{1}{J} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix},$$

where A_1 for example is the minor of a_1 in (24). (The equality refers to the nine elements individually, not merely to the determinants.) In the system S'

the directions of hyperosculation at P are accordingly given by

$$(27) \quad -(A_1 + B_1 Y' + C_1 Z') = \frac{A_2 + B_2 Y' + C_2 Z'}{Y'} = \frac{A_3 + B_3 Y' + C_3 Z'}{Z'}.$$

We wish to prove that the circles of the bundle which are determined by the three directions (23) at p coincide with the circles determined by the directions (27) at P. Since the line from p to P has been taken as the axis of abscissas, the first triple of circles will pass through point P in directions determined by the equations

$$(28) \quad -a_1 + a_2 Y' + a_3 Z' = \frac{-b_1 + b_2 Y' + b_3 Z'}{Y'} = \frac{-c_1 + c_2 Y' + c_3 Z'}{Z'},$$

which are found from (23) by simply replacing y' by $-Y'$ and z' by $-Z'$. It is easy to verify directly that (27) and (28) give the same three solutions; the calculation may be avoided, however, by observing that the equations are precisely of the form arising in the determination of the fixed points of a given collineation and of the inverse collineation, and such collineations of course have the same fixed points. We have then

THEOREM 7. *Consider any system S with property A and the reciprocal system S' defined above. Then the three hyperosculating circles at a given point p for the S curves are at the same time the hyperosculating circles for the S' curves at the corresponding point P.*

§ 4. Reciprocity of Natural Families.

Let us apply this result to the case where S is a natural family. The hyperosculating circles at p are then, according to property B, mutually orthogonal; the angles at the second point P are obviously the same as at p; hence the hyperosculating circles at P are mutually orthogonal, that is, the system S' will also possess property B. This gives our second fundamental result:

THEOREM 8. *The reciprocal of a natural family is also a natural family.*

Thus for each variation problem of the form

$$(1) \quad \int F(x, y, z)\, ds = \text{minimum},$$

assuming that J does not vanish identically, there is a second definitely related problem of the same form

$$(29) \quad \int \overline{F}(x, y, z)\, ds = \text{minimum}.$$

The problems may be termed reciprocal since the systems of extremals are reciprocal in the sense defined above.* We now prove

* It may happen in certain cases that the two systems coincide. Such cases arise when the transformation (17) is involutorial.

THEOREM 9. *Given the function F, the related function $\overline{F}$ may be determined by eliminations and differentiations.*

Letting L and $\overline{L}$ denote $\log F$ and $\log \overline{F}$ respectively, we find

$$\overline{L} = 2\int \frac{(\alpha - x)\,dx + (\beta - y)\,dy + (\gamma - z)\,dz}{(\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2}; \tag{31}$$

where α, β, γ denote the functions inverse to the functions

$$X = x + \frac{2L_x}{L_x^2 + L_y^2 + L_z^2},\quad Y = y + \frac{2L_y}{L_x^2 + L_y^2 + L_z^2},\quad Z = z + \frac{2L_z}{L_x^2 + L_y^2 + L_z^2}, \tag{32}$$

so that

$$x = \alpha(X, Y, Z), \qquad y = \beta(X, Y, Z), \qquad z = \gamma(X, Y, Z). \tag{32'}$$

The differential which appears in the determination of $\overline{L}$ will necessarily be exact in virtue of the reciprocity theorem proved above.

By actual integration we find

$$\overline{L}(X, Y, Z) = \log(L_x^2 + L_y^2 + L_z^2) - L. \tag{33}$$

In the right hand member, the variables x, y, z, after the differentiations, are to be replaced by the values given in (32'). The function $\overline{L}(x, y, z)$ may be obtained from the function $\log(L_x^2 + L_y^2 + L_z^2) - L$ by substituting for x, y, z, the functions $\alpha(x, y, z), \beta(x, y, z), \gamma(x, y, z)$, respectively.

§ 5. THE TRANSFORMATIONS T AND T_1. DARBOUX DEFORMATIONS.

The point transformation T connected with the general S system (7), by means of which the point p corresponds to the second vertex P of the bundle of osculating circles, according to the formulas

$$X = x + \frac{2\phi}{\phi^2 + \psi^2 + \chi^2}, \text{ etc.}, \tag{T}$$

is entirely general. If however the S system is a natural family, then T must have some peculiarity. What is the character of the transformation T in the case of a natural family? That is, what is the effect on T of the relations

$$\psi_z - \chi_y = 0, \qquad \chi_x - \phi_z = 0, \qquad \phi_y - \psi_x = 0?$$

To answer this we consider first another transformation

$$X_1 = x + \phi, \qquad Y_1 = y + \psi, \qquad Z_1 = z + \chi. \tag{T_1}$$

This associates with the point p a point P_1 which is easily shown to be the pole, with respect to a unit sphere about p as center, of the plane bisecting perpendicularly the line from p to P, that is, the plane containing the centers of

curvature. Otherwise stated, p, P, and P_1 are collinear and

$$\overline{pP} \cdot \overline{pP_1} = 2. \tag{34}$$

Thus each of the transformations T and T_1 uniquely determines the other. Hence any property of T_1 is virtually a property of T. We wish to prove

THEOREM 10. *For any natural family the associated transformation T_1 is of the* DARBOUX *type (defined below). Conversely, if T_1 is of this type, the corresponding S system will be natural.*

The category of transformations here described as of the DARBOUX type were met with by DARBOUX in the study of deformations of space.* For an arbitrary space deformation

$$x_1 = \Phi(x, y, z), \qquad y_1 = \Psi(x, y, z), \qquad z_1 = \chi(x, y, z), \tag{35}$$

there exist at each point three directions which are converted into parallel directions. This triplet will, at every point of space, be tri-rectangular when, and only when,

$$\Psi_z - \chi_y = 0, \qquad \chi_x - \Phi_z = 0, \qquad \Phi_y - \Psi_x = 0. \tag{36}$$

This is the criterion of a DARBOUX transformation.

Applying the test to T_1 we readily verify theorem 10. The property here proved is seen to be an equivalent of property B. Thus natural systems may be characterized completely, without introducing hyperosculating circles, as follows: †

The osculating circles at a given point must form a bundle (property A); and the associated transformation T_1, which is then induced, must be such that the three directions at each point which are converted into parallel directions are tri-rectangular (property D).

It is obvious that the inverse of any DARBOUX transformation will also be of that type.‡ The reciprocity theory for natural systems is not however contained in this statement. In connection with any transformation T we consider, in addition to the inverse transformation $t = T^{-1}$, two other transformations T_1 and t_1. If T converts point p into point P, then T_1 converts p into P_1 and t_1 converts P into p_1, where P_1 and p_1 are two points on the line pP so situated that

$$\overline{pP} \cdot \overline{pP_1} = 2, \qquad \overline{Pp}\ \overline{Pp_1} = 2. \tag{37}$$

* Proceedings of the London Mathematical Society, 1900.

† The statement D is preferable to statement B in so far as the calculations to which it leads are simpler and more easily extended to higher dimensions. But B is on the whole better since it renders the reciprocity relation intuitive. The three special directions described in D are not in general the same as those appearing in B.

‡ The totality of DARBOUX transformations do not form a group with respect to the ordinary multiplicative combination. They do, however, form a group with respect to *addition* defined as follows: if one transformation converts P into P_1, and a second converts P into P_2, then the *sum* converts P into P_3, such that the vector PP_3 equals the sum of the vectors PP_1 and PP_2. Analytically, $x_3 = x_1 + x_2 - x$, etc.

Usually, of these four transformations, any one of which determines the other three, none is of the DARBOUX type. The reciprocity theory in § 4 amounts to the following

THEOREM 11. *When T_1 is of the Darboux type so also is t_1.*

For this means, according to theorem 10, that whenever S is a natural system so also is the reciprocal system S'.

It may be shown that the only case in which the transformation T connected with a natural system is of the DARBOUX type is that in which the function L satisfies an equation of the form

$$L_x^2 + L_y^2 + L_z^2 = \text{function of } L. \tag{38}$$

This means that the surfaces $L = \text{constant}$ must be parallel. Hence in the dynamical interpretation the lines of force must be straight. We have thus a characterization of the rectilinear conservative fields of force.

§ 6. VELOCITY SYSTEMS.

The most general system with property A is represented by differential equations of the form

$$y'' = (\psi - y'\phi)(1 + y'^2 + z'^2), \qquad z'' = (\chi - z'\phi)(1 + y'^2 + z'^2), \tag{7}$$

and thus involves three arbitrary functions. Only in the case where these functions are the partial derivatives of the same function is the system a natural one. We now point out a dynamical problem that leads to the general type (7).

Consider a particle (of unit mass) moving in any field of force, the components of the force being ϕ, ψ, χ. The equations of motion are then

$$\ddot{x} = \phi(x, y, z), \qquad \ddot{y} = \psi(x, y, z), \qquad \ddot{z} = \chi(x, y, z). \tag{39}$$

If the initial position and the initial velocity are given the motion is determined. If only the initial position and direction of motion are given, the osculating plane will be determined but the radius of curvature r will depend for its value on the initial speed v. Hence, in addition to the usual formula

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2,$$

there must be a formula expressing v^2 in terms of x, y, z, y', z', r. This is furnished by the familiar equation

$$v^2 = rN \tag{40}$$

where N denotes the (principal) normal component of the force, so that

$$N^2 = \phi^2 + \psi^2 + \chi^2 - \frac{(\phi + y'\psi + z'\chi)^2}{1 + y'^2 + z'^2}. \tag{40'}$$

The result may be written in the two forms

$$v^2 = \frac{(\psi - y'\phi)(1 + y'^2 + z'^2)}{y''} = \frac{(\chi - z'\phi)(1 + y'^2 + z'^2)}{z''}. \tag{41}$$

In the actual trajectory v varies from point to point. If now we replace v^2 in this result by some constant, say $1/c$, the resulting equations may be written

$$\begin{aligned} y'' &= c(\chi - y'\phi)(1 + y'^2 + z'^2), \\ z'' &= c(\chi - z'\phi)(1 + y'^2 + z'^2). \end{aligned} \tag{42}$$

The curves satisfying these differential equations — they are not in general trajectories — we define as *velocity curves*. In a given field of force there are ∞^5 trajectories and ∞^5 velocity curves.* If c is given we have ∞^4 velocity curves. In particular if c (and hence v) is taken to be unity, our equations become precisely (7).

THEOREM 12. *Any system of ∞^4 curves possessing property A, that is, any system* (7), *may be regarded as the totality of velocity curves corresponding to unit velocity in some (uniquely defined) field of force.*

Only when the field is conservative do the velocity systems for each value of v (or c) become natural systems. The trajectories also are in this case made up of ∞^1 natural families, one for each value of h the energy constant; but the two sets of natural families are distinct. The determination of a velocity system in one conservative field is equivalent to the determination of a trajectory system in another conservative field, and vice versa. We find in fact the following

THEOREM 13. *If two conservative fields with work functions W_1 and W_2 satisfy the relation*†

$$W_2 = ae^{\frac{2W_1}{v_0^2}} - h \tag{43}$$

then the ∞^4 velocity curves for the speed v_0 in the first field coincide with the ∞^4 trajectories for the constant of energy h in the second field.‡

HALPHEN (Comptes rendus, vol. 84) showed that the only forces in space for which all the ∞^5 trajectories are plane curves are the central and parallel forces. We inquire when the system of ∞^4 velocity curves is made up of plane curves, and find that the requisite condition is

$$\Sigma(\psi - y'\phi)(\chi_x + y'\chi_y + z'\chi_z) \equiv 0, \tag{44}$$

where the summation refers to circular permutation of ϕ, ψ, χ. This leads to precisely the same set of partial differential equations that appear in HALPHEN's problem. Hence we have

* The properties of a complete system of ∞^5 velocity curves are analogous to, but distinct from, those of a complete system of trajectories.

† We note that if W_1 is left unaltered and v_0 varied, W_2 takes quite distinct forms. The ∞^1 velocity systems in a given field do not constitute the complete system of ∞^5 trajectories in any field whatever.

‡ It is seen that the two fields have the same equipotential surfaces and therefore the same lines of force. (Central fields therefore correspond to central fields.)

THEOREM 14. *The ∞^4 curves of a system of type* (7), *that is, of a velocity system, will be plane curves when and only when the vector field defined by* ϕ, ψ, χ, *is central or parallel.*

These two corollaries are of interest:

If in a conservative field the trajectories corresponding to one value of the constant of energy h are plane curves, the same will be true for all values of h, and the potential will vary according to some function of the distance from either a fixed point or a fixed plane.

The only cases in which the extremals connected with the variation problem

$$\int F(x, y, z)\, ds = \text{minimum} \tag{1}$$

are plane curves are those in which F is expressible as a function of a single argument of the type *

$$c_0(x^2 + y^2 + z^2) + c_1 x + c_2 y + c_3 z + c_4.$$

§ 7. CONFORMAL TRANSFORMATIONS.

If an arbitrary point transformation is applied to space, the ∞^4 curves of a natural family will not usually be converted into curves constituting a natural family. We now prove the following

THEOREM 15. *The only point transformations which convert every natural family into a natural family are those belonging to the conformal group.*

This may be proved by means of the differential equations of the family, but appears more readily from the variation problem (1). If an integral of this form is to be converted into one of the same form, the transformation must convert Fds into an expression of the same type, say $F_1 ds$. Therefore ds must be converted into a multiple of itself; that is, the equation $ds = 0$, defining the minimal lines, must be invariant. This is a familiar characteristic property of conformal transformations. Hence our result is established.

Natural systems have properties A and B. It is of interest to notice that property A by itself is conformally invariant. More precisely we have

THEOREM 16. *If a conformal transformation is applied to any system possessing property A, that is, any system of the type* (7), *the new system will also be of that type. No other point transformations leave type* (7) *invariant.*†

The proof of the last part of the theorem is most easily obtained by noticing that the equations of any system (7), namely,

$$y'' = (\psi - y'\phi)(1 + y'^2 + z'^2), \qquad z'' = (\chi - z'\phi)(1 + y'^2 + z'^2), \tag{7}$$

* The surfaces $F =$ constant must then be either concentric spheres or parallel planes. The result is made most concrete by the optical interpretation in which F equals ν the index of refraction. The author has applied this result to determine all isotropic media in which the rays of light are circular.

† It may be shown that, for any non-conformal transformation, there exists a unique system of type (7), i. e., a unique velocity system, which is converted into a velocity system.

admit as particular integrals the solutions of the set

$$1 + y'^2 + z'^2 = 0, \qquad y'' = 0, \qquad z'' = 0.$$

The latter equations obviously represent the minimal straight lines of space. No other solutions are common to all systems (7). Hence the ∞^3 minimal straight lines are the only lines (straight or curved) included in every system with property A. If therefore a transformation is to leave type (7) invariant, it must leave this part, common to all systems of that type, invariant. Hence no other than conformal transformations can fulfill our conditions.

The first part of the theorem may be proved synthetically.* The ∞^{10} conformal transformations of space convert circles into circles, and bundles of circles into bundles of circles. Hence if the original system possesses property A, the same will be true of the transformed system.

If the system possesses property B in addition to A, the same will obviously be true after conformal transformation; for hyperosculating circles will remain such and the tri-rectangular aspect will be preserved.

§ 8. Two-dimensional Results.

We have seen that natural systems of curves constitute a special species of the general class made up of all velocity systems. In the case of the plane, our systems contain ∞^2 curves (instead of ∞^4 as in space); the velocity type becomes

$$y'' = (\psi - y'\phi)(1 + y'^2), \tag{45}$$

involving two arbitrary functions $\phi(x, y)$, $\psi(x, y)$; and the natural type is

$$y'' = (L_y - y' L_x)(1 + y'^2), \tag{46}$$

corresponding to the restriction

$$\phi_y - \psi_x = 0. \tag{46'}$$

Another noteworthy case arises in the theory of isogonal trajectories. If for a given system of ∞^1 curves all the isogonals are constructed, the resulting system of ∞^2 curves has an equation of the form

$$y'' = (\Omega_x + y'\Omega_y)(1 + y'^2). \tag{47}$$

This corresponds to a different restriction, namely,

$$\phi_x + \psi_y = 0. \tag{47'}$$

Theorem 17. *Included in the velocity type, in the case of two dimensions,*

* This part of the reasoning would be insufficient in the case of two dimensions, since it would be valid only for the six-parameter group of Möbius circular transformations. Direct calculation, however, establishes the truth of the result for the infinite conformal group of the plane. *Cf.* American Journal of Mathematics, vol. 27 (1906), p. 213.

*are two noteworthy species: the natural and the isogonal. The former arises when the vector field ϕ, ψ is lamellar, the latter when the field is solenoidal.**

For each velocity system S there is a reciprocal system S'. Our discussion shows that when S is natural so also is S', and SCHEFFER's work shows that when S is isogonal S' is also. However, systems S for which

$$J \equiv \text{jacobian}\left(x + \frac{2\phi}{\phi^2 + \psi^2}, y + \frac{2\psi}{\phi^2 + \psi^2}\right)$$

vanishes identically do not give rise to a reciprocal system.†

The only systems which are common to the two species described are those which satisfy both restrictions

$$\phi_y - \psi_x = 0, \qquad \phi_x + \psi_y = 0.$$

The function L and also the function Ω will then be harmonic. The type arising will then consist of velocity systems in Laplacian fields, or, what is equivalent, of the isogonals of an isothermal system.‡

It follows that if S is composed of the isogonals of an isothermal system the same will be true of the reciprocal system S'.

Every natural system in the plane may be regarded as the conformal representation of the geodesics of some surface. From this point of view we may state

THEOREM 18. *A system of ∞^2 plane curves, one through each point in each direction, may be regarded as the conformal representation of the system of geodesics on some surface when and only when (A) the circles of curvature constructed at a given point form a pencil and (B) the two hyperosculating circles existing in such a pencil are orthogonal.*

The reciprocity theorem takes the following form: With every conformal representation of any surface σ upon a plane there is associated a conformal representation of another surface σ' upon the plane, defined by the fact that the two systems in the plane which depict the two systems of geodesics are reciprocal systems S and S'.

If the first conformal representation is given, S and hence S' is determined;

* Condition (46′) means that the curl vanishes, while (47′) means that the divergence vanishes.

† The existence of such exceptional cases is apparently not brought out in SCHEFFER's theory. Reciprocal isogonal systems may be regarded, from a certain point of view, as images of surfaces conjugate with respect to a null system. The exceptional systems then appear as images of developable surfaces.

‡ Cf. the author's note, *Isothermal systems in dynamics*, Bulletin of the American Mathematical Society, vol. 14 (1908), pp. 169–172. The Laplacian fields in the plane (i. e., those with a harmonic potential) are distinguished from all others by the fact that each of the ∞^1 systems of velocity curves is a linear two-parameter system.

the corresponding surface σ' is then fixed except for isometric deformation and homothetic transformation.*

The curvature transformation Γ.

Any conformal transformation of the plane converts the general type of velocity system

$$y'' = (\psi - y'\phi)(1 + y'^2) \tag{45}$$

into itself;† furthermore, each species, the natural (or geodesic) and the isogonal, is separately invariant.

We now show that *the following transformation*

$$x_1 = x, \qquad y_1 = y, \qquad y_1' = -\frac{1}{y'}, \qquad y_1'' = -\frac{y''}{y'^3} \tag{Γ}$$

interchanges the two species. Applying it in fact to the general type (45), we find that the new differential equation is of the same form, with ϕ and ψ replaced respectively by ϕ_1 and ψ_1, where

$$\phi_1 = -\psi, \qquad \psi_1 = \phi.$$

Hence if the original field ϕ, ψ is solenoidal the new field ϕ_1, ψ_1 will be conservative, and vice versa. We state the result as

THEOREM 19. *The isogonal system*

$$y'' = (\Omega_x + y'\Omega_y)(1 + y'^2),$$

which consists of all the isogonals of the simple system

$$y' = \tan^{-1}\Omega,$$

is converted by Γ *into the natural system*

$$y'' = (\Omega_y - y'\Omega_x)(1 + y'^2),$$

which may be obtained as the conformal representation of the geodesics of the surface whose squared element of length is

$$ds^2 = e^{2\Omega}(dx^2 + dy^2).$$

The transformation Γ is obviously not a contact transformation. It may be defined as the operation by means of which any curvature element (x, y, y', y'') is converted into another curvature element $(x, y, -1/y', -y''/y'^3)$ obtained from the first by rotating it about its own point (x, y) through a right angle, its curvature being preserved.‡

* If σ is a developable surface the same is true of σ'. It is possible that there are other cases in which the two surfaces are isometrically equivalent.

† Cf. American Journal of Mathematics, vol. 28 (1906), p. 213.

‡ The operation Γ^2 converts each curvature element into its opposite $(x, y, y', -y'')$. This process leaves each of our two species invariant.

With each natural system there is associated, by means of Γ, a definite isogonal system, and vice versa. Hence every property of the one species gives indirectly a property of the other. Thus for an isogonal system, while it is not true that the two hyperosculating circles in each pencil are orthogonal, this will be true for the associated system derived by means of Γ.

We are now in position to complete the theory of isogonals. The CESARO-SCHEFFERS theorem asserts that for such a system the circles of curvature at a given point p pass through a second common point P. Construct the point P_1 on the segment pP such that $pP \cdot pP_1 = 2$, and also the point P_2 obtained by rotating the segment pP_1 about p through a right angle. The transformation T_2 which converts every point p into the point P_2 determined in this way will then be of the DARBOUX type,* that is, will be such that the two lineal elements at each point which are converted into parallel elements are perpendicular to each other.

THEOREM 20. *Hence a system of ∞^2 plane curves will be a system of isogonals when, and only when, in addition to the* CESARO-SCHEFFERS *property, it satisfies the requirement that the induced transformation T_2 (defined above) is of the* DARBOUX *type.*

Thus if a transformation T is given the velocity system thereby determined will be a natural or an isogonal system in those cases where the related T, or T_2 transformation is of the DARBOUX type.

To what extent may the members of a system of specified type (velocity, natural, or isogonal) be taken arbitrarily? If a net of curves is given, i. e., two curves through each point of the plane, then a velocity system may be constructed to which that net belongs. *A natural system, like an isogonal system, is determined by any ∞^1 curves, one through each point of the plane (or the region considered), together with an arbitrary transversal curve (not tangent to any of the ∞^1 curves).* This result, which is fairly intuitive for the isogonal case, may be proved without difficulty by means of the CAUCHY existence theorem for partial differential equations.

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* The transformation T_1 associated with an isogonal system will be of the DARBOUX type only in the isothermal case. It may be shown that, for any isogonal system, T_1 has the following characteristic property: at any point the two lineal elements each of which is converted into a perpendicular element are perpendicular to each other.

PLANE FIELDS OF FORCE WHOSE TRAJECTORIES ARE INVARIANT UNDER A PROJECTIVE GROUP*

BY

GEORGE WILBER HARTWELL

A particle moving freely in a plane under the action of a force which depends only on the position of the particle has equations of motion of the form

$$m\frac{d^2x}{dt^2} = \phi(x, y), \qquad m\frac{d^2y}{dt^2} = \psi(x, y).$$

The functions $\phi(x, y)$ and $\psi(x, y)$ are assumed to be uniform and to possess first and second derivatives over the part of the plane considered. The case in which both are zero is excluded since then the field of force vanishes and the trajectories will consist only of the ∞^2 straight lines of the plane.

The particle may be projected from any position x_0, y_0, at any time t_0, and with any velocity, given by the direction y_0' and the speed v_0. By varying the four arbitrary constants x_0, y_0, y_0', v_0, all trajectories are obtained. However, since each trajectory can be described by starting from any one of its points with the proper velocity, each field of force gives rise to only a triply infinite system of curves.

The differential equation of such a triply infinite system of curves has the form†

$$y''' = Hy''^2 + Gy''$$

where

$$H = \frac{-3\phi}{\psi - \phi y'}, \qquad G = \frac{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2}{\psi - \phi y'}.$$

Any differential equation of this type is transformed into another of the same type by means of the collineations of the plane.‡ There are no other contact transformations which convert every system of trajectories into a system of trajectories.§ Although the type is invariant, each single equation of this type

* Presented to the Society November 28, 1908.

† E. KASNER, *The trajectories of dynamics*, these Transactions, vol. 7 (1906), pp. 401–424.

‡ Ibid., p. 420. The importance of collineations in dynamics was first indicated by APPELL, *De l'homographie en mécanique*, American Journal of Mathematics, vol. 12 (1890).

§ E. KASNER, *Note on the transformations of dynamics* (abstract), Bulletin of the American Mathematical Society, vol. 13 (1906), p. 157.

will not in general be invariant under any collineation. The object of this paper is to determine the plane fields of force that give rise to a triply infinite system of trajectories invariant under a continuous group of projective transformations.

In sections I–VI we determine the forms of the functions $\phi(x, y)$ and $\psi(x, y)$ such that the corresponding system of trajectories is invariant under some one of the projective groups completely tabulated by LIE.* In section VII the results of the preceding sections are collected. In section VIII the fields of force corresponding to groups of three or more parameters are considered in detail; the systems of trajectories are here given in finite form, and the cases indicated in which the trajectories are conics. It is of interest to note how the familiar forces of nature appear among those whose trajectories allow the largest groups.

I. ONE-PARAMETER GROUPS.

§ 1. $xp + ayq$, $a \neq 0, 1$ [35].†

Every one-parameter continuous group of point transformations contains one and only one infinitesimal transformation

$$\xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}, \qquad \text{or} \qquad \xi(x, y)p + \eta(x, y)q.$$

On the other hand each such infinitesimal transformation belongs to a single one-parameter group and consequently can be said to generate that group.‡

The increments given to x and y by such an infinitesimal transformation are

$$\delta x = \xi\delta t, \qquad \delta y = \eta\delta t.$$

The increments given to $y', y'', \cdots, y^{(n)}$ are determined by the relation §

$$\delta(dy^{(n-1)} - y^{(n)}dx) = 0.$$

The infinitesimal transformation $xp + ayq$, $a \neq 0, 1$, gives to x, y, y', y'', y''' the increments

$$\delta x = x\delta t, \quad \delta y = ay\delta t, \quad \delta y' = (a-1)y'\delta t, \quad \delta y'' = (a-2)y''\delta t, \quad \delta y''' = (a-3)y'''\delta t$$

We write the equation of our trajectories in the form

$$D \equiv y''' - Hy''^2 - Gy'' = 0.$$

If $D = 0$ is invariant under the group, then

$$\frac{\partial D}{\partial x}\delta x + \frac{\partial D}{\partial y}\delta y + \frac{\partial D}{\partial y'}\delta y' + \frac{\partial D}{\partial y''}\delta y'' + \frac{\partial D}{\partial y'''}\delta y''' = 0,$$

* LIE-SCHEFFERS, *Continuierliche Gruppen*, pp. 288–291.

† The number in brackets indicates the place of the corresponding group in LIE's list.

‡ By the group $\xi p + \eta q$ we shall mean the group generated by this infinitesimal transformation.

§ LIE-SCHEFFERS: *Geometrie der Berührungstransformationen*, p. 137.

by means of $D=0$ when δy, δx, etc., are replaced by their values.

Expanding this, we find

$$[xH_x+ayH_y+(a-1)y'H_{y'}+(a-1)H]y''^2+[xG_x+ayG_y+(a-1)y'G_{y'}+G]=0,$$

which must be satisfied identically. Hence

$$xH_x+ayH_y+(a-1)H_{y'}+(a-1)H=0, \tag{1}$$

$$xG_x+ayG_y+(a-1)G_{y'}+G=0. \tag{2}$$

The general solution of this set is

$$H=x^{1-a}\alpha\left(\frac{x^a}{y},\ x^{1-a}y'\right), \qquad G=\frac{1}{x}\beta\left(\frac{x^a}{y},\ x^{1-a}y'\right),$$

where α, β denote arbitrary functions of their arguments. But since H is of the form $-3\phi/(\psi-\phi y')$, we have

$$\frac{-3\phi}{\psi-\phi y'}=x^{1-a}\alpha\left(\frac{x^a}{y},\ x^{1-a}y'\right). \tag{3}$$

Placing $x^a/y=u$, $x^{1-a}y'=v$, we find

$$\frac{-3\phi}{x^{1-a}\psi-v\phi}=\alpha(u,v). \tag{4}$$

By hypothesis ϕ and ψ are functions of x and y alone. They do not contain y' and hence must be independent of v. The left-hand member of (4) can be put in the form $-3/(x^{1-a}\psi/\phi)-v$. Since this is a function of u and v, $x^{1-a}\psi/\phi$ must be a function of u alone. The most general forms of ϕ and ψ are then

$$\phi=F(x,y)F_1\left(\frac{x^a}{y}\right), \tag{5}$$

$$\psi=x^{a-1}F(x,y)F_2\left(\frac{x^a}{y}\right), \tag{6}$$

in which $F(x,y)$ is an unknown function to be determined from

$$G=\frac{\psi_x+(\psi_y-\phi_x)y'-\phi_y y'^2}{\psi-\phi y'}. \tag{7}$$

From (5), (6), and (7), we then have*

$$\frac{xF_x}{F}+xy'\frac{F_y}{F}+\frac{aF_2'-\dfrac{y}{x^a}F_2+\left(F_2-\dfrac{ax^a}{y}F_1\right)x^{1-a}y'+\left(\dfrac{x^a}{y}\right)^2(x^{1-a}y')^2F_1'}{\dfrac{y}{x^a}F_2-x^{1-a}y'F_1}=\beta\left(\frac{x^a}{y},x^{1-a}y'\right). \tag{8}$$

* By F_1', F_2' we shall always mean the first derivatives with respect to the single argument involved.

Since F_1, F_2, F_1', F_2' are functions of x^a/y alone, the third term of the left-hand member is a function of x^a/y, $x^{1-a}y'$. The other terms of the left-hand member must then be a function of the same arguments, say

$$\frac{xF_x}{F} + xy'\frac{F_y}{F} = \beta_1\left(\frac{x^a}{y}, x^{1-a}y'\right). \tag{9}$$

The function F, however, does not contain y'. Hence both xF_x/F and x^aF_y/F must be functions of x^a/y. We choose the form of these functions in the way most convenient to avoid integral signs as follows:

$$\frac{xF_x}{F} = a\frac{x^a}{y}\beta_2'\left(\frac{x^a}{y}\right), \qquad \frac{x^aF_y}{F} = \beta_3\left(\frac{x^a}{y}\right). \tag{10), (11}$$

From (10) we have

$$\log F = \beta_2\left(\frac{x^a}{y}\right) + f(y). \tag{12}$$

Hence, from (11) and (12),

$$-\frac{x^{2a}}{y^2}\beta_2'\left(\frac{x^a}{y}\right) + x^af'(y) = \beta_3\left(\frac{x^a}{y}\right).$$

We thus find

$$f'(y) = \frac{c}{y}, \qquad f(y) = \log c_1y^c,$$

$$F(x, y) = c_1y^c e^{\beta_2}.$$

Placing $c_1F_1e^{\beta_2} = F_3$ and $c_1F_2e^{\beta_2} = F_4$, we have as the final solution

$$\phi = y^cF_3\left(\frac{x^a}{y}\right), \qquad \psi = \frac{y^{c+1}}{x}F_4\left(\frac{x^a}{y}\right).$$

§ 2. $p + yq$ [36].

This one-parameter group gives to x, y, y', y'', y''' the infinitesimal increments

$$\delta x = \delta t, \qquad \delta y = y\delta t, \qquad \delta y' = y'\delta t, \qquad \delta y'' = y''\delta t, \qquad \delta y''' = y'''\delta t.$$

By the same course of reasoning as in the preceding case, we must have H and G such functions that they satisfy the two equations

$$H_x + yH_y + y'H_{y'} + H = 0, \tag{1}$$

$$G_x + yG_y + y'G_{y'} = 0. \tag{2}$$

Solving (1) and (2) we have

$$H = \frac{1}{y}\alpha\left(e^{-x}y, \frac{y'}{y}\right), \qquad G = \beta\left(e^{-x}y, \frac{y'}{y}\right). \tag{3), (4}$$

Therefore

$$\frac{-3\phi}{\frac{\psi}{y}-\phi\frac{y'}{y}}=\alpha\left(e^{-x}y,\frac{y'}{y}\right).$$

By a course of reasoning similar to that employed in the preceding case, we find

(5), (6) $\qquad \phi=F(x,y)\cdot F_1(e^{-x}y), \qquad \psi=F(x,y)\cdot F_2(e^{-x}y).$

From (4), (5), (6), and the relation connecting ϕ, ψ and G, we have

$$\frac{F_x}{F}+y'\frac{F_y}{F}+\frac{-e^{-x}yF'_2+(F_2+e^{-x}yF'_2+e^{-x}yF'_1)\frac{y'}{y}-e^{-x}y\left(\frac{y'}{y}\right)^2F'_1}{F_2-\frac{y'}{y}F_1}=\beta\left(e^{-x}y,\frac{y'}{y}\right).$$

The third term of the left-hand member is a function of $e^{-x}y$, y'/y, hence F_x/F and $y(F_y/F)$ must be a function of $e^{-x}y$. We choose these unknown functions in the following convenient form:

(7), (8) $\qquad \frac{F_x}{F}=-e^{-x}y\frac{\beta'_1(e^{-x}y)}{\beta_1(e^{-x}y)}, \qquad y\frac{F_y}{F}=\beta_2(e^{-x}y).$

Integrating (7), we have

(9) $\qquad \log F=\log\beta_1(e^{-x}y)+f(y).$

Then, from (8) and (9),

(10) $\qquad e^{-x}y\frac{\beta'_1(e^{-x}y)}{\beta_1(e^{-x}y)}+yf'(y)=\beta_2(e^{-x}y),$

$$f'(y)=\frac{c}{y}, \qquad f(y)=\log c_1y^c.$$

Hence

$$F(x,y)=c_1y^c\beta_1(e^{-x}y).$$

Changing the notation, we write our result in the form

$$\phi=y^cF_3(e^{-x}y), \qquad \psi=y^{c+1}F_4(e^{-x}y).$$

§ 3. $p+xq$ [37].

The conditions to be satisfied by H and G are here

(1), (2) $\qquad H_x+xH_y+H_{y'}=0, \qquad G_x+xG_y+G_{y'}=0.$

The solutions are

(3), (4) $\qquad H=\alpha(x^2-2y,x-y'), \qquad G=\beta(x^2-2y,x-y').$

The same method of reasoning used in the two preceding cases gives

$$\phi = F(x, y) \cdot F_1(x^2 - 2y), \tag{5}$$

$$\psi = xF(x, y) \cdot F_1(x^2 - 2y) + F(x, y) \cdot F_2(x^2 - 2y). \tag{6}$$

From (2), (5), (6) and the relation between G and ϕ and ψ, we find

$$\frac{F_x}{F} + y'F + \frac{F_1 + 2F_1' \cdot (x - y')^2 + 2(x - y')F_2'}{(x - y')F_1 + F_2} = \beta(x^2 - 2y, x - y'). \tag{7}$$

The third term of the left-hand member is a function of the two arguments $x^2 - 2y$ and $x - y'$. Then $F_x/F + y' \cdot F_y/F$ is some function of the same two arguments. We have then

$$\frac{F_x}{F} = \frac{2x\beta_1'(x^2 - 2y)}{\beta_1(x^2 - 2y)} + \beta_2(x^2 - 2y), \qquad \frac{F_y}{F} = -\frac{2\beta_1'(x^2 - 2y)}{\beta_1(x^2 - 2y)}. \tag{8), (9}$$

From (9), we have then

$$\log F = \log \beta_1(x^2 - 2y) + f(x). \tag{10}$$

From (8) and (10), we find that

$$f'(x) = \beta_2(x^2 - 2y).$$

Hence $f'(x) = c$, $f(x) = cx + \log c_1$. Therefore

$$F(x, y) = c_1 e^{cx} \beta_1(x^2 - 2y).$$

The final solution may be written in the form

$$\phi = e^{cx} F_3(x^2 - 2y),$$

$$\psi = xe^{cx} F_3(x^2 - 2y) + e^{cx} F_4(x^2 - 2y).$$

§ 4. $xp + yq$ [38].

The result may be written

$$\phi = y^c F_3\left(\frac{x}{y}\right), \qquad \psi = y^c F_4\left(\frac{x}{y}\right).$$

These functions can be obtained from the expressions for ϕ and ψ determined for the case of the group [35] by making $a = 1$ and changing the notation.

§ 5. q [39].

The conditions that $D = 0$ shall be invariant under this, the last of the one-parameter groups, are

$$H_y = 0, \qquad G_y = 0;$$

so that

$$H = \alpha(x, y'), \qquad G = \beta(x, y').$$

Since

$$\frac{-3\phi}{\psi-\phi y'}=\alpha(x, y'),$$

we must have

$$\phi=F(x, y)\cdot F_1(x), \qquad \psi=F(x, y)\cdot F_2(x).$$

These give

$$\frac{F_x}{F}+y'\frac{F_y}{F}+\frac{F_2'-y'F_1'}{F_2-y'F_1}=\beta(x, y').$$

The third term of the left hand member is a function of x and y'; hence

$$\frac{F_x}{F}=\frac{\beta_1'(x)}{\beta_1(x)}, \qquad \frac{F_y}{F}=\beta_2(x).$$

From these, we find

$$F(x, y)=c_1e^{cy}\beta_1(x).$$

The resulting field is

$$\phi=e^{cy}F_3(x), \qquad \psi=e^{cy}F_4(x).$$

II. Two-parameter groups.

§ 6. $p+xq, q$ [24].

In order to determine the forms of ϕ and ψ such that $D=0$ shall be invariant under the group generated by two infinitesimal transformations we will first determine the form of ϕ and ψ and hence H and G such that $D=0$ is invariant under one of the infinitesimal transformations; then impose the additional conditions on H and G necessary for invariance under the second.

We have already seen (§ 3) that when $D=0$ is invariant under group [37],* $\phi=e^{cx}F_3(x^2-2y)$ and $\psi=xe^{cx}F_3(x^2-2y)+e^{cx}F_4(x^2=2y)$ and hence

$$(1) \qquad H=\frac{-3F_3}{F_4+(x-y')F_3},$$

$$(2) \qquad G=\frac{F_3+cxF_3+xF_{3x}+F_{4x}+cF_4+(xF_{3y}+F_{4y}-cF_3-F_{3x})y'-F_{3y}y'^2}{F_4+(x-y')F_3}.$$

If $D=0$ is also invariant under q, then

$$(3), (4) \qquad H_y=0, \qquad G_y=0.$$

To be invariant under both groups, ϕ and ψ must be such functions that (1) and (2) are consistent with (3) and (4).

From (1) and (3) we have

$$(5) \qquad F_4F_{3y}-F_3F_{4y}=0,$$

* By group [37], etc., we mean throughout this paper the thirty-seventh, etc., group according to the numbering in Lie's list.

and this gives

(6) $$F_4 = f(x) \cdot F_3.$$

But F_4 and F_3 are functions of $x^2 - 2y$; hence $f'(x)$ must be a constant, that is,

(7) $$F_4 = c_1 F_3.$$

From (1) and (2) we have

(8) $$G = \frac{1 + cx + cc_1 - cy'}{c_1 + x - y'} + \frac{F_{3x}}{F_3} + y' \frac{F_{3y}}{F_3}.$$

Since F_3 is independent of y' we have, from (8) and (4),

(9), (10) $$\frac{\partial}{\partial y}\left(\frac{F_{3x}}{F_3}\right) = 0, \qquad \frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right) = 0.$$

Therefore

(11), (12) $$\frac{F_{3x}}{F_3} = \frac{f_1'(x)}{f_1(x)}, \qquad \frac{F_{3y}}{F_3} = f_2(x).$$

From (11) we find

(13) $$\log F_3 = \log f_1(x) + f_3(y),$$

and, from (12) and (13),

$$f_3'(y) = f_2(x) = -2c_2.$$

Therefore

$$f_3(y) = -2c_2 y + \log c_3,$$

$$\log F_3 = -2c_2 y + \log f(x) + \log c_3.$$

Bnt $\log F_3$ is a function of $x^2 - 2y$ alone. Therefore we must have

$$\log f(x) = c_2 x^2.$$

The forms of F_3, F_4 are thus found to be

$$F_3 = c_3 e^{c_2(x^2-2y)}, \qquad F_4 = c_1 c_3 e^{c_2(x^2-2y)}.$$

The corresponding field of force is

$$\phi = c_3 e^{c_2(x^2-2y)+cx}, \qquad \psi = (c_3 x + c_4) e^{c_2(x^2-2y)+cx} \qquad (c_4 = c_1 c_3).$$

§ 7. $p + yq, q$ [25].

Using the result under group [36], and imposing the extra conditions

(1), (2) $$H_y = 0, \qquad G_y = 0$$

we find

(3) $$F_3 = c_1 e^{-x} y F_4,$$

(4 $$G = \frac{(c+1)y'}{y} + \frac{c_1 e^{-x} y y'}{y - c_1 e^{-x} y y'} + \frac{F_{4x}}{F_4} + y' \frac{F_{4y}}{F_4}.$$

Then from (4) and (2), we have

$$(5) \qquad \frac{\partial}{\partial y}\left(\frac{F_{4x}}{F_4}\right) + y' \frac{\partial}{\partial y}\left(\frac{F_{4y}}{F_4}\right) - \frac{(c+1)y'}{y^2} = 0.$$

Since F_4 is independent of y', we have

$$(6), (7) \qquad \frac{\partial}{\partial y}\left(\frac{F_{4x}}{F_4}\right) = 0, \qquad \frac{\partial}{\partial y}\left(\frac{F_{4y}}{F_4}\right) = \frac{c+1}{y^2}.$$

Then, from (7),

$$(8) \qquad \frac{F_{4x}}{F_4} = \frac{f'(x)}{f(x)}, \qquad \log F_4 = \log f(x) + f_1(y).$$

From (7) we have

$$(9) \qquad \frac{F_{4y}}{F_4} = -\frac{c+1}{y} + f_2(x), \qquad \log F_4 = -(c+1)\log y + y f_2(x) + f_3(x).$$

Since the right-hand member of (8) consists of the sum of a function of x and one of y, and F_4 is itself a function of $e^{-x}y$, we must have

$$(10) \qquad f(x) = c_2 x \qquad \text{and} \qquad f_1(y) = -c_2 \log y.$$

Then from (9) and (10) we have $c_2 = c + 1$. Therefore

$$F_4 = \left(\frac{e^x}{y}\right)^{c+1}, \qquad F_3 = c_1\left(\frac{e^x}{y}\right)^{c}.$$

The field is

$$\phi = c_1 e^{cx}, \qquad \psi = e^{(c+1)x}.$$

§ 8. $q + xp$, xq [26].

We have shown (§ 2) that when $D = 0$ is invariant under the group $p + yq$, we have $\phi = y^c F_3(e^{-x}y)$, $\psi = y^{c+1}F_4(e^{-x}y)$. The group $q + xp$ can be obtained from the $p + yq$ by the interchange of x and y. Therefore our functions must be of the form

$$\phi = x^{c+1}F_4(e^{-y}x), \qquad \psi = x^c F_3(e^{-y}x).$$

This gives

$$(1) \qquad H = \frac{-3xF_4}{F_3 - xy'F_4},$$

$$(2) \qquad G = \frac{cF_3 + xF_{3x} + [(c+1)xF_4 - x^2F_{4x} + xF_{3y}]y' - x^2y'^2F_{4y}}{F_3 - xy'F_4}.$$

The additional conditions expressing invariance under xq are found to be

$$(3), (4) \qquad xH_y + H_{y'} = 0, \qquad xG_y + G_{y'} = 0.$$

From (1) and (3) we have

$$F_3 F_{4y} - F_4 F_{3y} + F_4^2 = 0. \tag{5}$$

Let $e^{-y}x = v$. Then $F_{4y} = -v \cdot dF_4 / dv$ and likewise for F_{3y}. Then (7) becomes

$$F_3 F_4' - F_4 F_3' - \frac{F_4^2}{v} = 0. \tag{6}$$

Integrating (6), we have

$$F_3 = -(\log c_1 v) F_4 = -F_4 \cdot \log c_1 e^{-y} x. \tag{7}$$

Then, from (2) and (7), we have

$$G = \frac{c}{x} + \frac{1}{xy' + \log c_1 e^{-y} x} + \frac{F_{4x}}{F_4} + y' \frac{F_{4y}}{F_4}. \tag{8}$$

From (4) and (8), it follows that

$$x \frac{\partial}{\partial y}\left(\frac{F_{4x}}{F_4}\right) + xy' \frac{\partial}{\partial y}\left(\frac{F_{4y}}{F_4}\right) + \frac{F_{4y}}{F_4} = 0. \tag{9}$$

Since F_4 is independent of y', this breaks up into

$$\frac{\partial}{\partial y}\left(\frac{F_{4y}}{F_4}\right) = 0, \qquad x \frac{\partial}{\partial y}\left(\frac{F_{4x}}{F_4}\right) + \frac{F_{4y}}{F_4} = 0. \tag{10), (11}$$

From (10), we have

$$\log F_4 = y f_1(x) + f(x). \tag{12}$$

Since F_4 is a function of $e^{-y}x$, we must have $f(x) = c_2 \log x +$ constant, and $f_1(x) = -c_2$, so that

$$F_4 = c_3 x^{c_2} e^{-c_2 y}.$$

Substituting this expression in (11), we find $c_2 = 0$, so that

$$F_4 = c_3, \qquad F_3 = -c_3 \log c_1 e^{-y} x = +c_3 y - c_3 \log c_1 x.$$

The field is

$$\phi = c_3 x^{c+1}, \qquad \psi = c_3 y x^c - c_3 x^c \log c_1 x.$$

§ 9. $xp + ayq$, xq [27].

From the first of these infinitesimal transformations we have, § 1,

$$\phi = y^c F_3\left(\frac{x^a}{y}\right), \qquad \psi = \frac{y^{c+1}}{x} F_4\left(\frac{x^a}{y}\right).$$

Here

$$(1)\quad H=\frac{-3xF_3}{yF_4-xy'F_3},$$

$$(2)\quad G=\frac{\frac{y}{x}F_{4x}-\frac{y}{x^2}F_4+\left(\frac{c+1}{x}F_4+\frac{y}{x}F_{4y}-F_{3x}\right)y'-\left(\frac{c}{y}F_3+F_{3y}\right)y'^2}{\frac{y}{x}F_4-y'F_3}.$$

To be invariant under the group xq, according to § 8, H and G must satisfy the conditions

(3), (4) $$xH_y+H_{y'}=0, \qquad xG_y+G_{y'}=0.$$

Equations (1) and (3) then give

$$(5)\quad y\frac{F_3F_{4y}-F_4F_{3y}}{F_3^2}+\frac{F_4}{F_3}-1=0.$$

Integrating, we find

(6), (7) $$y\frac{F_4}{F_3}=y+f(x), \qquad F_4=\left(1+\frac{f(x)}{y}\right)F_3.$$

Since F_4 and F_3 are functions of x^a/y, $[1+f(x)/y]$ must be a function of the same argument. Therefore

$$(8)\quad f(x)=c_1x^a, \qquad F_4=\left(1+\frac{c_1x^a}{y}\right)F_3.$$

From (8) and (2), together with (4), we have

$$(9)\quad \frac{c}{y}-\frac{cxy'}{y^2}+\frac{F_{3y}}{F_3}+x\frac{\partial}{\partial y}\left(\frac{F_{3x}}{F_3}\right)+xy'\frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right)=0.$$

Since F_3 is independent of y',

(10), (11) $$\frac{c}{y}+\frac{F_{3y}}{F_3}+x\frac{\partial}{\partial y}\left(\frac{F_{3x}}{F_3}\right)=0, \qquad x\frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right)-\frac{cx}{y^2}=0.$$

Putting $x^a/y=v$, we have

$$F_{3y}=-\frac{x^a}{y^2}\frac{dF_3}{dv}=-\frac{x^a}{y^2}F_3',$$

$$\frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right)=\frac{\partial}{\partial y}\left(-\frac{x^a}{y^2}\frac{F_3'}{F_3}\right)=\left(\frac{x^a}{y^2}\right)^2\frac{d}{dv}\left(\frac{F_3'}{F_3}\right)+\frac{2x^a}{y^3}\frac{F_3'}{F_3}.$$

Then (10) becomes

$$(12)\quad v^2\frac{d}{dv}\left(\frac{F_3'}{F_3}\right)+2v\frac{F_3'}{F_3}=c.$$

Therefore

$$\frac{F_3'}{F_3} = \frac{c}{v} + \frac{c_2}{v_2},$$

$$F_3 = c_3\left(\frac{x^a}{y}\right)^c e^{-\frac{c_2 y}{x^a}}. \tag{13}$$

From (13) and (10), we have $c_2(a-1)=0$ and hence $c_2=0$, since $a \neq 1$ by hypothesis. Therefore

$$F_3 = c_3\left(\frac{x^a}{y}\right)^c, \qquad F_4 = c_3\left(\frac{x^a}{y}\right)^c + c_1 c_3\left(\frac{x^a}{y}\right)^{c+1}.$$

The field is

$$\phi = c_3 x^{ac}, \qquad \psi = c_3 y x^{ac-1} + c_4 x^{ac+a-1}.$$

§ 10. xp, q [28].

The conditions that $D=0$ shall be invariant under xp are

$$xH_x - y'H_{y'} - H = 0, \qquad xG_x - y'G_{y'} + G = 0.$$

Omitting the discussion, we give the result

$$\phi = x^c F_3(y), \qquad \psi = x^{c-1}F_4(y).$$

Then we have

$$H = \frac{-3xF_3}{F_4 - xy'F_3},$$

$$G = \frac{\frac{c-1}{x}F_4 + (F_4' - cF_3)y' - xy'^2F_3'}{F_4 - xy'F_3}.$$

From the second generator, q, we have

$$H_y = 0, \qquad G_y = 0.$$

$$F_3'F_4 - F_4'F_3 = 0, \qquad F_3 = c_1 F_4.$$

The forms of the F functions are found to be

$$F_4 = c_3 e^{c_2 y}, \qquad F_3 = c_3 c_1 e^{c_2 y};$$

and the corresponding field is

$$\phi = c_3 x^c e^{c_2 y}, \qquad \psi = c_4 x^{c-1} e^{c_2 y}.$$

§ 11. p, q [29].

When $D=0$ is invariant under the group q (§ 5), we have

$$\phi = e^{cy}F_3(x), \qquad \psi = e^{cy}F_4(x).$$

The infinitesimal transformations p and q are equivalent since the transforma-

tion $x_1 = y$, $y_1 = x$ changes one into the other. Hence we must have also

$$\phi = e^{cx} F_5(y), \qquad \psi = e^{cy} F_6(y).$$

The final result is the field

$$\phi = c_3 e^{cx+cy}, \qquad \psi = c_4 e^{cx+cy}.$$

§ 12. q, xq [30].

From § 5, we have $\phi = e^{cy} F_3(x)$ and $\psi = e^{cy} F_4(x)$. Also

(1), (2) $$H = \frac{-3F_3}{F_4 - y'F_3}, \qquad G = cy' + \frac{F_4' - y'F_3'}{F_4 - y'F_3}.$$

That this may be invariant under the group xq, we must have, (§ 8),

(3), (4) $$xH_y + H_{y'} = 0, \qquad xG_y + G_{y'} = 0.$$

From (1) and (3) we have $F_3 = 0$, and from (2), (4) and (5) we have $c = 0$. Therefore

$$\phi = 0, \qquad \psi = F_4(x).$$

§ 13. xp, yq [31].

From § 10, we have

(1), (2) $$H = \frac{-3xF_3}{F_4 - xy'F_3}, \qquad G = \frac{\dfrac{c-1}{x} F_4 + (F_4' - cF_3)y' - xy'^2 F_3'}{F_4 - xy'F_3}.$$

The extra conditions, from yq, are

(3), (4) $$yH_y + y'H_{y'} + H = 0, \qquad yG_y + y'G_{y'} = 0.$$

From (1) and (3)

$$y(F_3'F_4 - F_4'F_3) + F_3F_4 = 0.$$

Integrating this we have

(5) $$F_4 = c_1 y F_3.$$

From (2), (4) and (5)

$$yF_3''F_3 - yF_3'^2 + F_3F_3' = 0.$$

Integration gives

$$F_3 = c_3 y^{c_2}.$$

The result is

$$\phi = c_3 x^c y^{c_2}, \qquad \psi = c_4 x^{c-1} y^{c_2+1}$$

§ 14. $xp + yq$, q [32].

From § 4. we have

$$\phi = y^c F_3\left(\frac{x}{y}\right), \qquad \psi = y^c F_4\left(\frac{x}{y}\right).$$

Also

$$(1),\ (2)\qquad H=\frac{-3F_3}{F_4-y'F_3},\qquad G=\frac{cy'}{y}+\frac{F_{4x}+(F_{4y}-F_{3x})y'-F_{3y}y'^2}{F_4-y'F_3}.$$

The additional requirements are

$$(3),\ (4)\qquad H_y=0,\qquad G_y=0.$$

Then (1) and (3) give

$$F_4\frac{\partial}{\partial y}F_3-F_3\frac{\partial}{\partial y}F_4=0.$$

But F_3 and F_4 are both functions of x/y. Therefore

$$(5)\qquad F_4=c_1F_3.$$

From (2), (4) and (5) we have

$$-\frac{cy'}{y^2}+\frac{F_{3xy}F_3-F_{3x}F_{3y}}{F_3^2}+y'\frac{F_{3yy}F_3-F_{3y}^2}{F_3^2}=0.$$

Since F_3 is independent of y', we have

$$(6),\ (7)\qquad \frac{F_{3xy}F_3-F_{3x}F_{3y}}{F_3^2}=0,\qquad -\frac{c}{y}+\frac{F_{3yy}F_3-F_{3y}^2}{F_3^2}=0;$$

and thus

$$(8),\ (9)\qquad \frac{F_{3y}}{F_3}=f(y),\qquad \frac{c}{y}+\frac{F_{3y}}{F_3}=f_1(x).$$

From these,

$$f(y)+\frac{c}{y}=f_1(x),$$

$$(10)\qquad f_1(x)=c_2,\qquad f(y)=c_2-\frac{c}{y}.$$

Then from (8) and (10) we have

$$\log F_3=c_2y-c\log y+f_2(x),\qquad F_3=\frac{1}{y^c}e^{c_2y+f_2(x)}.$$

But F_3 is a function of x/y and hence $f_2(x)$ must be such a function that the right-hand member is also a function of x/y. Therefore

$$f_2(x)=c\log x,\qquad c_2=0,$$

$$F_3=\left(\frac{x}{y}\right)^c,\qquad F_4=c_1\left(\frac{x}{y}\right)^c.$$

The force is found to be

$$\phi=x^c,\qquad \psi=c_1x^c.$$

§ 15. q, yq [33].

From § 5 we have $\phi = e^{cy}F_3(x)$, $\psi = e^{cy}F_4(x)$, and thus

(1), (2) $$H = \frac{-3F_3}{F_4 - y'F_3}, \qquad G = cy' + \frac{F_4' - F_3'y'}{F_4 - F_3y'}.$$

Invariance under the operator yq requires the two conditions

(3), (4) $$yH_y + y'H_{y'} + H = 0, \qquad yG_y + y'G_{y'} = 0.$$

From (1) and (3), we have $F_3F_4 = 0$; therefore $F_3 = 0$ or $F_4 = 0$, since we have excluded the case in which both ϕ and ψ vanish. If $F_3 = 0$, from (2) and (4) we have $c = 0$. If $F_4 = 0$, (2) and (4) again give $c = 0$. Therefore

$$\phi = 0, \psi = F_4(x) \qquad \text{or} \qquad \phi = F_3(x), \psi = 0.$$

§ 16. $xp + 2yq$, $p + xq$ [34].

When $D = 0$ is invariant under group [35], we have (§ 1)

$$\phi = y^c F_3\left(\frac{x^a}{y}\right), \qquad \psi = \frac{y^{c+1}}{x} F_4\left(\frac{x^a}{y}\right).$$

Therefore to make it invariant under the group $xp + 2yq$, we must have

$$\phi = y^c F_3\left(\frac{x^2}{y}\right), \qquad \psi = \frac{y^{c+1}}{x} F_4\left(\frac{x^2}{y}\right).$$

We may put ψ in the form

$$\psi = y^c x F_5\left(\frac{x^2}{y}\right).$$

Then

(1) $$H = \frac{-3F_3}{xF_5 - y'F_3},$$

(2) $$G = \frac{cy'}{y} + \frac{yF_5 + xyF_{5x} + (xyF_{5y} - yF_{3x})y' - yy'^2F_{3y}}{xyF_5 - yy'F_3}.$$

In order to have $D = 0$ invariant under the group [34] we must adjoin the two conditions (§ 3)

(3), (4) $$H_x + xH_y + H_{y'} = 0, \qquad G_x + yG_y + G_{y'} = 0.$$

From (1) and (3), we have

(5) $$x(F_{3x}F_5 - F_3F_{5x}) + x^2(F_{3y}F_5 - F_3F_{5y}) + F_3^2 - F_3F_5 = 0.$$

If we put $x^2/y = v$, (5) becomes

$$(2v - v^2)(F_3'F_5 - F_5'F_3) + F_3^2 - F_3F_5 = 0.$$

Integrating, we have

$$F_3 = \left(1 + \frac{c_1}{x}\sqrt{x^2 - 2y}\right) F_3. \tag{6}$$

From (2) and (6), we have

$$G = \frac{cy'}{y} + \frac{c_1 x - c_1 y' + \sqrt{x^2 - 2y}}{(x - y' + c_1\sqrt{x^2 - 2y})\sqrt{x^2 - 2y}} + \frac{F_{3x}}{F_3} + y'\frac{F_{3y}}{F_3}. \tag{7}$$

From (4) and (7), we have

$$\frac{c}{y} - \frac{cxy'}{y^2} + \frac{\partial}{\partial x}\left(\frac{F_{3x}}{F_3} + y'\frac{F_{3y}}{F_3}\right) + x\frac{\partial}{\partial y}\left(\frac{F_{3x}}{F_3} + y'\frac{F_{3y}}{F_3}\right) + \frac{F_{3y}}{F_3} = 0.$$

Since F_3 is independent of y'

$$\frac{c}{y} + \frac{\partial}{\partial x}\left(\frac{F_{3x}}{F_3}\right) + x\frac{\partial}{\partial y}\left(\frac{F_{3x}}{F_3}\right) + \frac{F_{3y}}{F_3} = 0, \tag{8}$$

$$-\frac{cx}{y^2} + \frac{\partial}{\partial x}\left(\frac{F_{3y}}{F_3}\right) + x\frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right) = 0. \tag{9}$$

Again let $x^2/y = v$. Then

$$\frac{\partial}{\partial x}\left(\frac{F_{3y}}{F_3}\right) = -\frac{2x}{y^2}\frac{F_3'}{F_3} - \frac{2x^3}{y^3}\frac{d}{dv}\left(\frac{F_3'}{F_3}\right); \qquad \frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right) = \frac{2x^2}{y^3}\frac{F_3'}{F_3} - \frac{x^4}{y^4}\frac{d}{dv}\left(\frac{F_3'}{F_3}\right).$$

Hence (9) can be written

$$-\frac{cx}{y^2} + 2\left(\frac{x^3}{y^3} - \frac{x}{y^2}\right)\frac{F_3'}{F_3} + \left(\frac{x^5}{y^4} - \frac{2x^3}{y^3}\right)\frac{d}{dv}\left(\frac{F_3'}{F_3}\right) = 0.$$

Dividing by x/y^2 and replacing x^2/y by v we have

$$c - 2(v-1)\frac{F_3'}{F_3} - (v^2 - 2v)\frac{d}{dv}\left(\frac{F_3'}{F_3}\right) = 0. \tag{10}$$

Integrating (10), we have

$$\log F_3 = c\log(v-2) + \frac{c_2}{2}\log\frac{v-2}{v} + \log c_3,$$

$$F_3 = c_3\left(\frac{x^2 - 2y}{y}\right)^c\left(\frac{x^2 - 2y}{x^2}\right)^{\frac{c_2}{2}}.$$

Then, from (8), we have $c_2 = 0$; accordingly we have

$$F_3 = c_3\left(\frac{x^2 - 2y}{y}\right)^c, \qquad F_3 = c_3\left(\frac{x^2 - 2y}{y}\right)^c + \frac{c_1 c_3}{x}\left(\frac{x^2 - 2y}{y}\right)^{c+\frac{1}{2}}.$$

The field is found to be

$$\phi = c_3(x^2 - 2y)^c, \qquad \psi = (x^2 - 2y)^c(c_3 x + c_4\sqrt{x^2 - 2y}).$$

III. Three-parameter groups.

§ 17. xq, q, p [13].

We proceed from two-parameter to three-parameter groups in the same way that we proceeded from one-parameter to two-parameter groups. We choose the ϕ and ψ corresponding to a two-parameter subgroup, determine the form of H and G, and then impose the condition for invariance under a third infinitesimal transformation.

It was found, § 12, that when $D=0$ is invariant under group [30] we have $\phi=0$ and $\psi=F_4(x)$, and hence

(1), (2) $$H=0, \qquad G=\frac{F_4'}{F_4}.$$

To the two-parameter group we adjoin the generator p making the group [13]. The extra conditions are

(3), (4) $$H_x=0, \qquad G_x=0.$$

Then from (2) and (4) we have $F_4=e^{cx}$. The resulting field is

$$\phi=0, \qquad \psi=e^{cx}.$$

§ 18. $p+xq, xp+2yq, q$ [14].

When $D=0$ is invariant under group [34] we have (§ 16) $\phi=c_3(x^2-2y)^c$, $\psi=(x^2-2y)^c(c_3x+c_4\sqrt{x^2-2y})$. Here

(1) $$H=\frac{-3c_3}{c_3(x-y')+c_4\sqrt{x^2-2y}},$$

(2) $$G=\frac{c_4(x-y')+c_3\sqrt{x^2-2y}}{[c_3(x-y')+c_4\sqrt{x^2-2y}]\sqrt{x^2-2y}}+\frac{2c(x-y')}{x^2-2y}.$$

To render $D=0$ invariant under the group q we must have in addition (§ 5),

(3), (4) $$H_y=0, \qquad G_y=0.$$

Then from (1) and (3) we have $c_3c_4=0$. If $c_3=0$, (2) and (4) give $c=-\frac{1}{2}$. If $c_4=0$, (2) and (4) give $c=0$. The field is therefore of one of the two forms

$$\phi=0, \quad \psi=c_4; \qquad \phi=c_3, \quad \psi=c_3x.$$

§ 19. $p+yq, q, xq$ [15].

We have found (§ 7) that when $D=0$ is invariant under group [25] $\phi=c_1e^{cx}$, $\psi=e^{(c+1)x}$. Then

(1), (2) $$H=\frac{-3c_1}{e^x-c_1y'}, \qquad G=\frac{(c+1)e^x-cc_1y'}{e^x-c_1y'}.$$

For invariance under the group xq, we have (§ 8) the conditions

(3), (4) $$xH_y + H_{y'} = 0, \qquad xG_y + G_{y'} = 0.$$

Then from (1) and (3) we have $c_1 = 0$; (2) and (4) are then consistent. Therefore

$$\phi = 0, \qquad \psi = e^{(c-1)x}.$$

§ 20. $p, q, xp + (y - x)q$ [16].

When $D = 0$ is invariant under group [29] we have (§ 11) $\phi = c_3 e^{cx+c_2y}$, $\psi = c_4 e^{cx+c_2y}$. Then

(1), (2) $$H = \frac{-3c_3}{c_4 - c_3 y'}, \qquad G = c + c_2 y'.$$

From the third generator of our group we find

(3) $$xH_x + (y - x)H_y - H_{y'} = 0,$$

(4) $$xG_x + (y - x)G_y - G_{y'} + G = 0.$$

From (2) and (4) we have $c - c_2 + c_2 y' = 0$, whence $c = c_2 = 0$. From (1) and (3) we have then $c_3 = 0$. The field is thus simply

$$\phi = 0, \qquad \psi = c_4.$$

§ 21. $xq, xp - yq, yp$ [17].

From § 9, $\phi = c_3 x^{ne}$, $\psi = c_3 y x^{nc-1} + c_4 x^{ne+a-1}$. If $a = -1$ group [27] becomes the group $xq, xp - yq$. Then we have

$$\phi = \frac{c_3}{x^c}, \qquad \psi = \frac{c_3 y}{x^{c+1}} + \frac{c_4}{x^{c+2}},$$

and

(1), (2) $$H = \frac{-3c_3x^2}{c_3xy + c_4 - c_3x^2y'}, \qquad G = \frac{-(c+2)c_4}{c_3xy + c_4 - c_3x^2y'} - \frac{c+1}{x}.$$

For invariance under the group yp we must have

(3) $$yH_x - y'^2 Hy' - 2y'H + 3 = 0,$$

(4) $$yG_x - y'^2 Gy' + y'G = 0.$$

From (1) and (3) we find $c_4 = 0$. Then from (2), (4) and (5) we have $c = -1$. The field is

$$\phi = c_3 x, \qquad \psi = c_3 y.$$

§ 22. $p, q, (a-1)xp + ayq$ [18].

From group [29] we have $\phi = c_3 e^{cx+c_2y}$, $\psi = c_4 e^{cx+c_2y}$ (§ 11), and thus

(1), (2) $$H = \frac{-3c_3}{c_4 - c_3 y'}, \qquad G = c + c_2 y'.$$

The further conditions are

(3) $$(a-1)xH_x + ayH_y + y'H_{y'} + H = 0,$$

(4) $$(a-1)xG_x + ayG_y + y'G_{y'} + (a-1)G = 0.$$

Then from (2) and (4) we have

$$y'c_2 + (a-1)c + (a-1)c_2 y' = 0.$$

Therefore $c = 0$, $c_2 = 0$ when $a \neq 0$; $c = 0$ when $a = 0$; $c_2 = 0$ when $a = 1$. From (1) and (3) we have $c_3 c_4 = 0$; so that $c_3 = 0$ or $c_4 = 0$. The results are

$$\left.\begin{aligned} \phi &= 0, & \psi &= c_4 \\ \phi &= c_3, & \psi &= 0 \end{aligned}\right\} \text{ when } a \neq 0, 1;$$

$$\left.\begin{aligned} \phi &= 0, & \psi &= c_4 e^{c_2 y} \\ \phi &= c_3 e^{c_2 y}, & \psi &= 0 \end{aligned}\right\} \text{ when } a = 0;$$

$$\left.\begin{aligned} \phi &= 0, & \psi &= c_4 e^{cx} \\ \phi &= c_3 e^{cx}, & \psi &= 0 \end{aligned}\right\} \text{ when } a = 1.$$

§ 23. $xp + ayq, xq, q$ [19].

From group [27], $\phi = c_3 x^{ac}$, $\psi = c_3 y x^{ac-1} + c_4 x^{ac+a-1}$, and

(1) $$H = \frac{-3c_3 x}{c_3 y + c_4 x^a - c_3 x y'},$$

(2) $$G = \frac{c_4 a x^a}{c_3 x y + c_4 x^{a+1} - c_3 x^2 y'} + \frac{ac-1}{x}.$$

For invariance under the group q, we must have (§ 5)

(3), (4) $$H_y = 0, \qquad G_y = 0.$$

From (1) and (3) we find $c_3 = 0$. Therefore

$$\phi = 0, \qquad \psi = c_4 x^{ac+a-1}.$$

§ 24. yq, xp, q [20].

From the result for group [31], we have, § 13,

(1) $$H = \frac{-3c_3 x}{c_4 y - c_3 x y'},$$

(2) $$G = \frac{c}{x} + \frac{c_2 y'}{y} + \frac{c_4(xy' - y)}{c_4 xy - c_3 x^2 y'}.$$

The further conditions are

$$H_y = 0, \qquad G_y = 0.$$

The resulting fields are found to be

$$\phi = c_3 x^c, \ \psi = 0; \qquad \phi = 0, \ \psi = c_4 x^{c-1}.$$

§ 25. $xp + yq, q, p$ [21].

From group [32] we have (§ 14) $\phi = x^c$, $\psi = c_1 x^c$, and thus

$$H = \frac{-3}{c_1 - y'}, \qquad G = \frac{c}{x}.$$

The extra conditions give $c = 0$; hence $\phi = 1$, $\psi = c_1$.

§ 26. q, xq, yq [22].

From group [30] we have, § 12, $\phi = 0$, $\psi = F_4(x)$, and thus

$$H = 0, \qquad G = \frac{F_4'}{F_4}.$$

The third operator, yq, imposes no extra restrictions. Therefore

$$\phi = 0, \qquad \psi = F_4(x).$$

§ 27. $p + xq, xp + 2yq, (x^2 - y)p + xyq$ [23].

From the group [34], we have

$$\phi = c_3(x^2 - 2y)^c, \qquad \psi = (c_3 x + c_4 \sqrt{x^2 - 2y})(x^2 - 2y)^c,$$

and hence

$$(1) \qquad H = \frac{-3c_3}{c_3(x - y') + c_4 \sqrt{x^2 - 2y}},$$

$$(2) \qquad G = \frac{c_4(x - y') + c_3 \sqrt{x^2 - 2y}}{(c_3(x - y') + c_4 \sqrt{x^2 - 2y}) \sqrt{x^2 - 2y}} + \frac{2c(x - y')}{x^2 - 2y}.$$

The additional equations are

$$(3) \quad (x^2 - y)H_x + xyH_y + (y - xy' + y'^2)H_{y'} + 2y'H - xH - 3 = 0,$$

$$(4) \quad (x^2 - y)G_x + xyG_y + (y - xy' + y'^2)G_{y'} + 2xG - y'G + 3 = 0.$$

From (1) and (3) we have $c_3 = c_4 = 0$. There is then no field of force whose trajectories are invariant under the group [23].

IV. Four-parameter groups.

§ 28. $p, q, xq, axp + yq, a \neq \frac{1}{2}$ [7].

The system of trajectories is invariant under group [13] when we have (§ 17) $\phi = 0, \psi = e^{cx}$, and thus

(1), (2) $$H = 0, \qquad G = c.$$

The further conditions, due to the infinitesimal transformation $axp + yq$, are

(3) $$axH_x + yH_y + (1-a)y'H_{y'} + (1-a)H = 0,$$

(4) $$axG_x + yG_y + (1-a)y'G_{y'} + aG = 0.$$

From (2) and (4), we find $ac = 0$. The results are

$$\phi = 0,\ \psi = 1,\ a \neq 0; \qquad \phi = 0,\ \psi = e^{cx},\ a = 0.$$

§ 29. $q, xq, xp + 2yq, p$ [8].

When $a = 2$ group [19] becomes the three-parameter group $q, xq, xp + 2yq$. In this case we have $\phi = 0, \psi = c_4 x^{2c+1}$, and

$$H = 0, \qquad G = \frac{2c+1}{x}.$$

Invariance under the operator p is easily found to require $2c + 1 = 0$. Therefore the field is

$$\phi = 0, \qquad \psi = c_4.$$

§ 30. p, q, xq, xp [9].

From group [13] we have (§ 17) $\phi = 0, \psi = e^{cx}$, and

$$H = 0, \qquad G = c.$$

Invariance under xp requires the conditions

$$xH_x - y'H_{y'} - H = 0, \qquad xG_x - y'G_{y'} + G = 0.$$

We find $c = 0$ and thus

$$\phi = 0, \qquad \psi = 1.$$

§ 31. xp, yq, xq, yp [10].

In order to determine ϕ and ψ so that $D = 0$ is invariant under group [10] we take the values of ϕ and ψ corresponding to the group [31], impose first the condition for invariance under the group xq, and then the condition for invariance under the group yp.

From group [31] we have (§ 13) $\phi = c_3 x^c y'^2$, $\psi = c_4 x^{c-1} y'^{2+1}$, and hence

(1), (2) $$H = \frac{-3c_3 x}{c_4 y - c_3 xy'}, \qquad G = \frac{c}{x} + \frac{c_2 y'}{y} + \frac{c_4(xy' - y)}{c_4 xy - c_3 x^2 y'}.$$

The operator xq imposes the conditions (§ 8)

(3), (4) $$xH_y + H_{y'} = 0, \qquad xG_y + G_{y'} = 0.$$

From (1) and (3) we have $c_3(c_3 - c_4) = 0$, so that $c_3 = 0$ or $c_3 = c_4$. If $c_3 = 0$, (2) and (4) give $c_2 = -1$. If $c_3 = c_4$, then (2) and (4) give $c_2 = 0$. Therefore

(5) $$\phi = 0, \qquad \psi = c_4 x^{c-1},$$

(6) $$\phi = c_3 x^c, \qquad \psi = c_3 x^{c-1} y.$$

We must now add the conditions due to yp (§ 21), namely,

(7), (8) $$yH_x - y'^2 H_{y'} - 2y'H + 3 = 0, \qquad yG_x + y'^2 G_{y'} + y'G = 0.$$

When ϕ and ψ are defined by (5) we have

(9), (10) $$H = 0, \qquad G = \frac{c-1}{x}$$

and when ϕ and ψ are defined by (6) we have

(11), (12) $$H = \frac{-3x}{y - xy'}, \qquad G = \frac{c-1}{x}.$$

Equations (7) and (9) are not consistent, hence (5) does not give a solution. Equations (7) and (11) are consistent, and (8) and (12) give $c = 1$. Therefore

$$\phi = c_3 x, \qquad \psi = c_3 y.$$

§ 32. xp, yq, q, p [11].

From group [20] (§ 24) either $\phi = c_3 x^c$, $\psi = 0$ or $\phi = 0$, $\psi = c_4 x^{c-1}$ and hence

$$H = \frac{3}{y'}, \qquad G = \frac{c}{x}, \qquad \text{or} \qquad H = 0, \qquad G = \frac{c-1}{x}.$$

The operator p requires $H_x = 0$, $G_x = 0$; from which we find

$$\phi = c_3, \quad \psi = 0; \qquad \phi = 0, \quad \psi = c_4.$$

§ 33. q, yq, xp, xq [12].

Under group [20], formulas (1), (2), (3), (4) (§ 32), give us the forms of H and G. The additional equations, due to xp, are

$$xH_y + H_{y'} = 0, \qquad xG_y + G_{y'} = 0.$$

The only solution is found to be

$$\phi = 0, \qquad \psi = c_4 x^{c-1}.$$

V. Five-parameter groups.

§ 34. $xp - yq, yp, xq, p, q$ [4].

Under group [17], we have (§ 21) $\phi = c_3 x$, $\psi = c_3 y$, and hence

$$H = \frac{-3x}{y - xy'}, \qquad G = 0.$$

The operator p requires

$$H_x = 0, \qquad G_x = 0.$$

The conditions are not consistent, hence there is no field of force whose trajectories are invariant under group [4].

§ 35. $xp - yq, yp, xq, x^2p + xyq, xyp + y^2q$ [5].

Under group [17], we have (§ 34)

$$H = \frac{-3x}{y - xy'}, \qquad G = 0.$$

The operator $x^2p + xyq$ gives

$$x^2 H_x + xyH_y + (y - xy')H_{y'} - xH = 0,$$

$$x^2 G_x + xyG_y + (y - xy')G_{y'} + 2xG + 3 = 0.$$

Hence no field of force exists.

§ 36. q, yq, xq, xp, p [6].

Under group [12], $\phi = 0$, $\psi = c_4 x^{c-1}$, and hence

$$H = 0, \qquad G = \frac{c - 1}{x}.$$

The operator p requires that $c = 1$. Hence the field is

$$\phi = 0, \qquad \psi = c_4.$$

VI. Groups of more than five parameters.

The remaining groups are two of six parameters

$$p, q, xp, yp, xq, yq \text{ [2]},$$

$$xp, yp, xq, yq, x^2p + xyq, xyp + y^2q \text{ [3]},$$

and the entire eight-parameter group

$$p, q, xp, yp, xq, yq, x^2p + xyq, xyp + y^2q \text{ [1]}.$$

Each of these contains as a subgroup either group [4] or group [5], which do not give rise to fields. Hence no field of force * corresponds to the present groups.

* Except, of course, the trivial field $\phi = 0$, $\psi = 0$, whose trajectories are straight lines.

VII. Table of the fields obtained.

The preceding results, simplified by projective reduction, and arranged in the order of Lie's list, are collected in the following table. The groups omitted, namely, 1, 2, 3, 4, 5, and 23, do not give rise to any solutions.

Lie's No.	Group.	ϕ	ψ
6	$q,\ yq,\ xq,\ xp,\ p$	0	1
7	$p,\ q,\ xq,\ axp+yq,\ a\neq\frac{1}{2}$ $\begin{cases} a\neq 0 \\ a=0 \end{cases}$	0 0	1 e^x
8	$q,\ xq,\ xp+2yq,\ p$	0	1
9	$p,\ q,\ xq,\ xp$	0	1
10	$xp,\ yq,\ xq,\ yp$	x	y
11	$xp,\ yq,\ q,\ p$	0	1
12	$q,\ yq,\ xp,\ xq$	0	x^c
13	$xq,\ q,\ p$	0	e^x
14	$p+xq,\ xp+2yq,\ q$	0 1	1 x
15	$p+yq,\ q,\ xq$	0	e^x
16	$p,\ q,\ xp+(y-x)q$	0	1
17	$xq,\ xp-yq,\ yp$	x	y
18	$p,\ q,\ (a-1)xp+ayq$ $\quad a\neq 0,\ 1$	1 0	0 1
	$\quad a=0$	0 e^y	e^y 0
	$\quad a=1$	0 e^x	e^x 0
19	$xp+ayq,\ xq,\ q$	0	x^{ac}
20	$yq,\ xp,\ q$	x^c 0	0 x^c
21	$xp+yq,\ q,\ p$	1	1
22	$q,\ xq,\ yq$	0	$F(x)$
24	$p+xq,\ q$	$e^{c_2(x^2-2y)+cx}$	$(c_1x+1)e^{c_2(x^2+2y)+cx}$
25	$p+yq,\ q$	e^{cx}	$e^{(c+1)x}$
26	$q+xp,\ xq$	x^{c+1}	$x^c(y-\log c_1x)$
27	$xp+ayq,\ xq$	x^{ac}	$x^{ac-1}(y+c_1x^a)$
28	$xp,\ q$	$x^{c_1}e^{c_2y}$	$x^{c_1-1}e^{c_2y}$
29	$p,\ q$	$e^{c_2x+c_1y}$	$c_3e^{c_2x+c_1y}$
30	$q,\ xq$	0	$F(x)$
31	$xp,\ yq$	$x^{c_1}y^{c_2}$	$c_\lambda x^{c-1}y^{c_2+1}$

LIE'S No.	Group.	ϕ	ψ
32	$xp+yq,\ q$	x^c	c_1x^c
33	$q,\ yq$	0	$F(x)$
		$F_1(x)$	0
34	$xp+2yq,\ p+xq$	$(x^2-2y)^c$	$(x^2-2y)^c(c_1\sqrt{x^2-2y}+x)$
35	$xp+ayq \quad a\neq 0,1$	$y^cF\left(\frac{x^a}{y}\right)$	$\frac{y^{c+1}}{x}F_1\left(\frac{x^a}{y}\right)$
36	$p+yq$	$y^cF(e^{-x}y)$	$y^{c+1}F_1(e^{-x}y)$
37	$p+xq$	$e^{cx}F(x^2-2y)$	$xe^{cx}F(x^2-2y)$ $+e^xF_1(x^2-2y)$
38	$xp+yq$	$y^cF\left(\frac{x}{y}\right)$	$y^cF_1\left(\frac{x}{y}\right)$
39	q	$e^{cy}F(x)$	$e^{cy}F_1(x)$

VIII. TRAJECTORIES OF THE FIELDS OF FORCE CORRESPONDING TO GROUPS OF THREE OR MORE PARAMETERS.

Among the fields of force whose trajectories are invariant under projective groups of three or more parameters, there are only eight projectively distinct types; these are given in the following table together with the largest group admitted.

No.	Group.		ϕ	ψ
6	$q,\ yq,\ xq,\ p,\ xp$		0	1
7	$p,\ q,\ xq,\ axp+yq$	$a=0$	0	e^c
10	$xp,\ xq,\ yp,\ yq$		x	y
12	$q,\ xq,\ yq,\ xp$		0	x^c
14	$p+xp,\ xp+2yq,\ q$		1	x
18	$p,\ q,\ (a-1)xp+ayq$	$a\neq 0$	0	e^y
20	$xp,\ yq,\ q$		x^c	0
22	$q,\ xq,\ yq$		0	$F(x)$

The differential equations of the trajectories of these eight types, together with their complete solutions, are given below. The calculations require only quadratures.

Group [6]:

$$y'''=0, \qquad y=\alpha x^2+\beta x+\gamma.$$

Group [7]:

$$y'''=y', \qquad y=\alpha e^x+\beta x+\gamma.$$

Group [10]:

$$y'''=\frac{-3xy''^2}{y-xy'}, \qquad \alpha x^2+\beta xy+\gamma y^2=1.$$

Group [12]:

$$y'''=\frac{cy''}{x}, \qquad y=\alpha x^{c+2}+\beta x+\gamma.$$

Group [14]:

$$y'''=\frac{3y''^2}{y'-x}+\frac{y''}{x-y'},$$

$$4x^4-48x^2y-16\alpha x^3+4x^2(\beta+4\alpha^2)+96\alpha xy+144y^2$$
$$-2x(4\alpha\beta+\gamma)-24\beta y+\beta^2-\alpha\gamma=0.$$

Group [18]:

$$y'''=y'y'', \qquad \gamma e^y=\sec^2(\alpha x+\beta).$$

Group [20]:

$$y'''=\frac{3y''^2}{y'}+\frac{cy''}{x}, \qquad y+\gamma=\int\frac{dx}{\sqrt{\alpha x^{c+1}+\beta}}.$$

Group [22]:

$$y'''=\frac{F'(x)}{F(x)}y'', \qquad y=\alpha\int\int F(x)\,dx\,dx+\beta x+\gamma.$$

From these results we see that the trajectories invariant under groups [7] and [18] are transcendental curves; those invariant under [6], [10], [12] and [14] are algebraic; while those invariant under [20] and [22] may be either.

Of especial interest are those forces in which the trajectories are a triply infinite system of conics. This is obviously the case for the trajectories invariant under groups [6] and [10]. For the first, the system consists of parabolas with vertical axes; and for the second, of the conics with a common center.

When $c=-3$ or $-\frac{3}{2}$ the trajectories invariant under [12] are conics given by the equations

(1) $$xy=\beta x^2+\gamma x+\alpha,$$

(2) $$(\alpha_1 y+\beta_1 x+\gamma_1)^2=x.$$

Equation (1) is that of the system of hyperbolas whose centers are on the y-axis and which have the y-axis as a common asymptote, a system which may easily be projected into the system of vertical parabolas. The projection $x=(x_1+iy_1)/(x_1-iy_1)$, $y=1/(x_1-iy_1)$, which transforms the line at infinity and the y-axis into the two minimal lines through the origin, transforms (2), since that is the equation of all parabolas tangent to the y-axis, into the equation of the conics with the origin as a common focus. The force is then newtonian attraction.

When $c = -3$ the trajectories, under [20] have the equation

$$y^2 + \beta_1 x^2 + \alpha_1 y + \gamma_1 = 0.$$

This gives the system of conics with centers on the y-axis and axes parallel to the coördinates axes, projectively equivalent to the system under group [10]. For $c = 0$ the trajectories invariant under the group [22] obviously become the parabolas with parallel axes.

No new cases are found under group [22]. It is known from the investigations of BERTRAND, HALPHEN, and APPELL that there exist only three projectively distinct types where the trajectories are conic sections: *First*, parabolas with parallel axes, generated by a constant parallel force, and allowing a five-parameter group. *Second*, conics with a common center, generated by a central force varying directly as the distance, and allowing a four-parameter group. *Third*, conics with a common focus, generated by a central force varying inversely as the square of the distance, and allowing a four-parameter group.

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ON THE ORDER OF PRIMITIVE GROUPS*

BY

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At the end of a memoir on primitive groups in the first volume of the Bulletin of the Mathematical Society of France,† JORDAN announced the following theorem:

Let q be a positive integer less than 6, p any prime number greater than q; the degree of a primitive group G that contains a substitution of order p on q cycles (without including the alternating group) cannot exceed $pq + q + 1$.

The proof of this theorem for $q = 1$ JORDAN published,‡ as well as that for the case of $q = 2$;† but for the values 3, 4, and 5 of q, no proofs have yet been published.

It is here shown to be possible to replace the above theorem by the following, which gives in part a closer limit:

Let q be an integer greater than unity and less than 5; p any prime greater than $q + 1$; then the degree of a primitive group which contains a substitution of order p that displaces pq letters (not including the alternating group) cannot exceed $pq + q$. When p is equal to $q + 1$, the degree cannot exceed $pq + q + 1$.

Before taking up the proof of this theorem, a series of general theorems on transitive groups must be established.

THEOREM I.§ *Let $s_1, s_2, \cdots$, be certain substitutions of order p (p an odd prime number) and of degree not greater than pq (q any number less than p) which generate a transitive group G. A number (λ) of these substitutions generate an intransitive subgroup I_1. A substitution s of order p and of degree not greater than pq can always be found in G, which connects letters of any given transitive constituent of I_1 with letters of some other transitive constituent of I_1.*

Suppose that the theorem is not true. Take a particular transitive con-

* Presented to the Society (San Francisco), September 29, 1906.

† C. JORDAN, Bulletin de la Société mathématique de France, vol. 1 (1873), pp. 175–221.

‡ C. JORDAN, ibid., vol. 1 (1873), pp. 40–71.

§ Cf. a related theorem in the writer's paper, *On the Primitive Groups of Class Ten*, American Journal of Mathematics, vol. 28 (1904), pp. 226–236.

stituent of $I_1 = \{s_1, s_2, \cdots, s_\lambda\}$ in the letters $a_1, a_2, \cdots$. From all the powers of all the conjugates under G of $s_1, s_2, \cdots$, choose a substitution $s_{\lambda+1}$ which connects one of the letters $a_1, a_2, \cdots$ with a new letter, and which, of all the substitutions of this totality that connect letters $a_1, a_2, \cdots$ with new letters $\alpha_1, \alpha_2, \cdots$, has the fewest new letters in the cycles with letters $a_1, a_2, \cdots$. There may be a number of such substitutions. Now consider the group $I_2 = \{I_1, s_{\lambda+1}\}$. If no substitution of the series $s_1, s_2, \cdots$ connects the extended set $a_1, a_2, \cdots$ with another transitive set of I_2, we take $s_{\lambda+2}$ as we did $s_{\lambda+1}$, and continue in this way until we have an intransitive subgroup I_{e-1} and a substitution s_e which connects the extended set $a_1, a_2, \cdots$ with some other set of I_{e-1}. It is now essential to consider closely the substitution s_{e-1} by which we pass from I_{e-2} to I_{e-1}. Let the letters of the first set of I_{e-2} be $a_1, a_2, \cdots, a_\kappa$, and let the remaining letters of I_{e-2}, which may or may not form a single transitive set, be denoted by $b_1, b_2, \cdots$; we shall speak of the letters a and the letters b.

Letters α, new to I_{e-2}, are connected with letters a by s_{e-1}. Since s_{e-1} is of prime order, any power of s_{e-1} connects a's and α's. If s_{e-1} has two α's in any one cycle, a number x may be chosen so that in s_{e-1}^x these two new letters are adjacent. Then unless s_{e-1}^x replaces all the k letters a by α's, one of the generators of the group $s_{e-1}^{-x} I_{e-2} s_{e-1}^x$ will connect a's and α's and displace fewer α's than does s_{e-1}, contrary to hypothesis. Suppose that s_{e-1}^x replaces every a by an α, but has two a's in the same cycle:

$$s_{e-1}^x = (a_1 \alpha_1 \cdots a_2 \alpha_2 \cdots) \cdots.$$

We may choose y so that

$$s_{e-1}^{xy} = (a_1 a_2 \cdots \alpha_1 \alpha_2 \cdots) \cdots,$$

and proceed as before. Finally, if in s_{e-1} all the cycles containing a letter a displace $p-1$ α's, then s_{e-1} must have at least p cycles, contrary to hypothesis. Hence s_{e-1} has not two letters α in any cycle. We may write

$$s_e = (a_\mu \beta \cdots) \cdots;$$

where β is a new letter followed by a b in some power, s_{e-1}^z say, of s_{e-1}, and μ is arbitrary. Now s_{e-1}^z has two letters a, as $a_\mu a_{\mu+1}$, adjacent in one of its cycles, so that

$$s_{e-1}^{-z} s_e s_{e-1}^z = (a_{\mu+1} b \cdots) \cdots.$$

Hence the theorem as stated is true.

THEOREM II.* *If the group $\{I_1, s\}$ is transitive, there is in it a substitution*

* Theorems II and IV were added by the author in March, 1908, after the presentation of the paper to the Society.

s' of order p which connects letters of the given set a of I_1 with other letters of I_1, and which is not of higher degree than that one of the substitutions $s_1, s_2, \cdots, s_\lambda$ which is of highest degree; nor does s' displace more new letters α than does s.

The generators of I_1 are $s_1, s_2, \cdots, s_\lambda$. Let the degree of s_λ be not lower than that of $s_1, s_2, \cdots,$ or $s_{\lambda-1}$. Suppose $\{I_1, s\}$ transitive and $s = (a_1 b_1 \cdots)\cdots$ of higher degree than s_λ. Now the totality of substitutions conjugate to $s_1, s_2, \cdots, s_\lambda$ generate an invariant subgroup H of $\{I_1, s\}$. If H is intransitive its transitive sets form systems of imprimitivity in $\{I_1, s\}$, and each is of degree p at least. But $\{I_1, s\}$ is generated by substitutions of order p and degree less than p^2, which cannot permute systems with so many as p letters in each of them. Then H is transitive, includes I_1, and consequently contains a substitution, of order p and of degree not greater than the degree of s_λ, which replaces one of the letters a by a letter b. Since this new s is found in $\{I_1, s\}$ it does not displace more new letters α than the former substitution s.

THEOREM III. *Let s be a substitution of the series $s_1, s_2, \cdots$ which connects transitively a given set a of I_1 (as before defined) with some other set of I_1. Let s be one of the substitutions which connect with the letters a the minimum number of new letters α. Moreover let it be assumed that no power of s replaces every letter of the set a by letters not belonging to the set. When these conditions are satisfied, s has not more than one new letter α to a cycle.*

We first remark that s may not have two new letters α adjacent in any cycle. For we may apply theorem I to that transitive constituent of $\{I_1, s^{-1} I_1 s\}$ which includes the letters a and letters b and infer that some generator of order p and of degree not greater than pq of the transformed group $s^{-1} I_1 s$ would have a letter a and a letter b in the same cycle and would displace fewer letters α than does s.

In the next place s may not have two letters α in the cycle $(a_1 b_1 \cdots)$. A certain power s^x certainly has two α's adjacent, and hence by the preceding paragraph it is impossible that any a should be followed by a b. Also no b is followed by an a, for then we should have $s^{-x} = (a'b' \cdots \alpha_2 \alpha_1 \cdots)$. Hence $s^x = (a'\alpha' \cdots b'\alpha'' \cdots)\cdots$ and $s^{xy} = (a'b' \cdots \alpha'\alpha'' \cdots)\cdots$, again in contradiction with the preceding result. Hence in any power of s a b either precedes or follows an a.

Finally s may not have two α's in any cycle. If in $s^{\pm x}$ two α's are adjacent, from $s^{\pm x} I_1 s^{\mp x}$ we may take a substitution of order p and of degree not greater than pq which connects letters a and b and displaces fewer than the minimum number of new letters α.

THEOREM IV. *A substitution s of order p and of degree not greater than pq (q less than p) can always be found in $\{s_1, s_2, \cdots\}$, which connects some two transitive sets of I_1 and which introduces at most q new letters.*

Let $s_1, s_2, \cdots$ and I_1 be defined as before and let the transitive constituents of I_1 displace the letters

$$\begin{aligned} a&: \quad a_1, a_2, \cdots, a_\kappa; \\ b'&: \quad b'_1, b'_2, \cdots, b'_{\kappa'}; \\ &\cdots\cdots\cdots \\ b^\nu&: \quad b^\nu_1, b^\nu_2, \cdots, b^\nu_{\kappa^\nu}; \end{aligned}$$

respectively. We know that in the group generated by $s_1, s_2, \cdots$, there are substitutions of order p and of degree not greater than pq which have in the same cycle letters from two or more of the sets $a, b', \cdots, b^\nu$ of I_1. From all such substitutions let us select those which displace the least possible number (μ) of letters α new to I_1, and call them $\sigma_1, \sigma_2, \cdots$. We wish to prove that the number μ is not greater than q.

Let s be any one of the substitutions $\sigma_1, \sigma_2, \cdots$, and let $s = (a_1 b'_1 \cdots) \cdots$, say. The theorem has already been proved if no power of s replaces every a by a letter not an a. Then it may be assumed that some power of each of the substitutions $\sigma_1, \sigma_2, \cdots$, has this property for each pair of transitive sets of I_1 that are joined in any cycle. Now suppose that s connects transitively the $\rho + 1$ sets $a, b', b'', \cdots, b^\rho$. Each of the sets $b', b'', \cdots, b^\rho$ has this property as well as a. For if it were not true for b^i, say, let b^i be taken for a and then apply theorem IV, according to which some substitution of order p and of degree not greater than pq connects b^i with some other set of I_1 and introduces at most q new letters. Then all the letters of these $\rho + 1$ sets are displaced by s; from which it follows, if we apply theorem II to this first transitive constituent of $\{I_1, s\}$, that no new letters α are present in the cycles of s which involve $a, b', \cdots, b^\rho$.

Suppose that s^x has two new letters adjacent in some cycle:

$$s^x = (a_1 b'_1 \cdots) \cdots (\alpha_1 \alpha_2 \cdots) \cdots.$$

Now from the transformed groups $s^{\pm x} I_1 s^{\mp x}$ we may infer that letters a can be followed in s^x only by letters b' and the letters b' can be preceded only by letters a, or in other words a b' never follows a b'. Again, if one b' is followed by an a, every b' is followed by an a. Then no b' is followed by an a, since p by hypothesis is an odd prime. It should be remarked that ρ is less than q, since each set of I_1 involves at least p letters, and the letters $a, b', \cdots, b^\rho$ fill up at most $q - 1$ cycles of s, and in consequence $\rho + 1$ is less than p. Now if letters b'' follow the b's, every b' is followed by a b'', no b'' may follow a b'', no b' may follow a b'', and if an a follows a b'', p is necessarily a multiple of 3. Then must the b'''s be followed by letters b''', while b''' may not be followed by letters b''', b'' or b', nor may it be followed by a's without making p a multiple of 4; and

so on. Finally p would be equal to $\rho+1$, but we have seen that $\rho+1$ is less than p. Hence no cycle of s displaces two letters α.

THEOREM V. *Let P be a cyclic group of degree pq and of order p. The largest group G on the same letters that transforms P into itself and that contains no operator of order p with less than q cycles is of order $p(p-1)(q!)$.*

The largest possible group on these letters, of which G is a subgroup, is of order $p^q(p-1)(q!)$. If G contains a substitution of order p which is not a power of a substitution of P, it certainly contains substitutions of order p and of degree less than pq. Hence the order of G is not greater than $p(p-1)(q!)$. Since we can in all cases construct a group of this order that shall transform G into itself by setting up a simple isomorphism between q metacyclic groups and adjoining substitutions which permute the q sets of intransitivity in every possible way, G is exactly of this order.

THEOREM VI. *The quotient group G/P is the direct product of a cyclic group of order $p-1$ and the symmetric group on q symbols.*

Since s, a generator of P, belongs to a set of $p-1$ conjugate substitutions, s is invariant in a subgroup I of order $p(q!)$. The quotient group I/P is symmetric. Again, s is transformed into all its powers by the substitutions of the metacyclic subgroup M of G. Let t be a substitution of I, and let u be a substitution of M that transforms s into s^x, x not congruent to unity, modulo p. Then $(t^{-1}ut)^{-1}s(t^{-1}ut)=s^x$ also. Since $\{I/P, M/P\}=G/P$, every operator of M/P is invariant in G/P. Hence the group G/P is a direct product.

COROLLARY. *When q is less than p, the substitutions of order p in the largest group G' of order $p^q(p-1)(q!)$ in which P is invariant generate an abelian group.*

THEOREM VII. *A primitive group of degree $2p+k$ $(p>k>2)$, which does not include the alternating group, cannot contain a substitution of order p and of degree $2p$.*

Let there be given a primitive group G which contains a substitution

$$A=(a_1a_2\cdots a_p)(b_1b_2\cdots b_p).$$

Since p is greater than 3, G can contain no substitution of order and of degree p.* There is a second substitution B, similar to A, in the transitive (because invariant) subgroup generated by the conjugates of A, which connects the two cycles of A, and displaces at most two new letters. Hence G *contains a transitive subgroup $H=\{A, B\}$ of degree not greater than $2p+2$.*

*C. JORDAN, Bulletin de la Société mathématique de France, vol. 1 (1873), pp. 40–71.

It may now be shown that if H is included in a primitive group of degree greater than $2p+2$, it is included in a group of degree $2p+3$ which is at least doubly transitive. If H is primitive this is certainly the case.* If H is of degree $2p$ and imprimitive, it cannot have two systems of p letters each, because it is generated by two substitutions of order p. If it has p systems of two letters each, it is contained in a doubly transitive group of degree $2p+3$, if not in one of degree $2p+1$. If H is of degree $2p+1$ it is certainly primitive. Finally let H be of degree $2p+2$. If the two letters left fixed by A belong to two different systems of imprimitivity, each system is of degree $p+1$, but that is not possible in $\{A, B\}$. Then the two letters left fixed belong to the same system and H has $p+1$ systems of two letters each. Since A determines this system by the two letters left fixed, the choice of systems can be effected in only one way, in other words, H is simply imprimitive with respect to a system of two letters. Hence *this group H must lead to a doubly transitive group of degree* $2p+3$. Then in G^{2p+3}, the subgroup $P=\{A\}$ is invariant in a subgroup I† which has the symmetric group of degree 3 as one constituent, which constituent we may call the group J.‡ For suppose P is transformed into itself by kp substitutions in the subgroup of G^{2p+3} which leaves two letters fixed. These kp substitutions involve only the letters of A. In the subgroup of G which leaves one letter fixed, since it is a transitive group of degree $2p+2$, P is invariant in a subgroup I_1 of order $2kp$ which has one constituent of order 2 on the two letters not displaced by A. Finally in G itself P is transformed into itself by $6kp$ substitutions and J is a transitive group of degree 3 and of order 6, a non-abelian group; while the quotient group of the largest group in which P may be invariant, involving only the letters of A, and containing no substitutions of order p and lower degree, is abelian.

It is well known that if $p=3$, the degree of G cannot exceed 9 and that there exists a primitive group G^9_{1512} of class 6 containing the substitution $(a_1a_2a_3)(b_1b_2b_3)$.

THEOREM VIII. *A primitive group of degree* $3p+k\,(p \geqq k > 3)$, *which does not include the alternating group, cannot contain a substitution of order p and of degree* $3p$.

In the first place G has no substitution of order p and of degree p or $2p$.

If theorem IV be twice applied, it is seen that G contains a transitive group H of degree not higher than $3p+6$ generated either by two (when the degree

* *On Multiply Transitive Groups*, these Transactions, vol. 7 (1906), pp. 499–508. This article will be used freely in what follows without further citation.

† *A Note on Transitive Groups*, Bulletin of the American Mathematical Society, vol. 13 (1906), p. 20.

‡ This notation, I for the largest subgroup of G in which $P=\{A\}$ is invariant, and J for the constituent on the letters left fixed by A, will be used throughout the remainder of this paper.

does not exceed $3p+3$) or by three similar substitutions of order p. In case the degree of H does not exceed $3p+3$ it will next be shown that G includes a doubly transitive subgroup of degree $3p+4$ whenever the degree of G exceeds $3p+3$. This is obviously true if the degree of H is $3p$, $3p+1$, or $3p+2$. If H is of degree $3p+3$, consider the three letters left fixed by $A=(a_1 a_2 \cdots a_p)(b_1 b_2 \cdots b_p)(c_1 c_2 \cdots c_p)$; one of them cannot belong to different systems of imprimitivity from the other two, and moreover a system of three letters may be chosen in but one way. Hence, in this case also, H leads to a doubly transitive group G^{3p+4}. Consider now the subgroup of G^{3p+4} which leaves two letters fixed. It is of order kp, say, and has perhaps one constituent of degree 2. Going back to G^{3p+4}, in it $P=\{A\}$ is invariant in a subgroup of order $12kp$ or $6kp$ which has a constituent on 4 letters which is doubly or triply transitive, that is, includes the tetraedral group. But the tetraedral group is not a subgroup of the quotient group with respect to P of the group of theorem V. Hence if the degree of G is greater than $3p+3$, the degree of H is also greater than $3p+3$.

Let H be of degree $3p+4$. It may not be imprimitive because A can only permute systems of three letters. Now if H is primitive, we know that it is simply transitive, and that the subgroup leaving one letter fixed is of degree $3p+3$, with two constituents.* One constituent cannot be of degree $2p+3$, nor can the other be of degree $p+3$ without making J tetraedral. If the larger constituent is of degree $2p+2$, or if the smaller is of degree $p+2$ (the only cases that remain) then in this intransitive subgroup of G the subgroup in which P is invariant has certainly a constituent of degree 2, so that in G itself I has a constituent of order 8 on 4 letters which involves a transposition, and hence is the octic group. But in the direct product of the symmetric group of order 6 and a cyclic group of order $p-1$ there is no octic subgroup.

Again H may be assumed to be of degree $3p+5$. It is primitive. If G^{3p+5} is doubly transitive, the subgroup leaving one letter fixed is transitive and is in turn primitive, for A in case of imprimitivity either permutes systems of three letters or leaves fixed systems of p or more letters. Just as before, this primitive group of degree $3p+4$ cannot exist. Now assume that G^{3p+5} is a simply transitive primitive group. The subgroup leaving one letter fixed has just two constituents, since $\{A, B\}$ has already a constituent of degree $2p+2$ at least. One constituent may be of degree $2p+2$, so that there will have to be a constituent of degree 5 and order 10 in I. This is impossible unless G contains a subgroup of order p^2, and $p=5$. In this case the two constituents are of degree 12 and 7. But a primitive group of degree 20 cannot contain a substitution of order 7 without including the alternating group. If there is a

*C. JORDAN, *Traité des Substitutions* (1870), p. 284; G. A. MILLER, Proceedings of the London Mathematical Society, vol. 28 (1897), p. 533.

constituent of degree $2p+3$ or $p+3$, that constituent is alternating or symmetric, and in the first case G certainly contains substitutions of order p of degree p or $2p$. In the second case, when p is greater than 5, the largest subgroup of G in which P is invariant has a transitive constituent of degree 5 which involves a cyclic substitution of order 3, that is, includes the alternating group of order 60; but this is impossible. When $p=5$, G contains a substitution of order 11, and hence must be alternating. This completes the consideration of H if it is of degree $3p+5$.

Let H be of degree $3p+6$. It cannot be doubly transitive, for then its transitive subgroup of degree $3p+5$ yields to the preceding analysis. If it is a simply transitive primitive group, its intransitive subgroup of degree $3p+5$ has just two constituents. Since $\{A, B\}$ has one constituent of degree $2p+2$ and the other of degree $p+1$, the largest subgroup of H in which P is invariant involves a transposition. But if p is greater than 5 this group has a transitive constituent of degree 6, whereas no subgroup of the direct product of the symmetric group of order 6 and a cyclic group of order $p-1$ can be written as a transitive group on 6 letters which involves a transposition. This remark holds as well when H is imprimitive. If $p=5$, and if H is primitive, H contains a substitution of order 7 involving not more than two cycles. If H is imprimitive, the subgroup leaving one letter fixed may contain no substitution of order 5 and degree 20, for such a substitution would not permute in the proper manner systems of three letters each. Then P is a Sylow subgroup, and again the subgroup of H in which it is invariant would have a constituent of degree 6 involving a transposition. This completes the proof of theorem VIII.

THEOREM IX. *A primitive group of degree* $4p+k\,(p \geqq k > 4)$, *which does not include the alternating group, cannot contain a substitution of order* p *and degree* $4p$.

By theorem IV, G contains a transitive subgroup H of degree not greater than $4p+12$ generated by two, three or four substitutions similar to A.

If H is an imprimitive group, its degree is $4p+2n$, where $n=0, 1, \cdots,$ or 6. If n is 3 or 5 the $2p+n$ systems of two letters each are permuted according to an alternating group, so that G is of class 6 at most. The primitive groups of class 6 are known and none is of higher degree than 10. For the same reason systems of two letters, when $n=4$ or 6, may not be permuted according to a primitive group. Now H cannot be of degree $4p+12$, because $\{A, B, C\}$ has a transitive constituent of degree $3p+6$ in which the group J is transitive and contains a transposition while having the quotient group with respect to an invariant operator a subgroup of the symmetric group on three symbols. This is true even for $p=5$. If H is of degree $4p+8$ it can have no subgroup of order p^2, for then H would include substitutions of order p hav-

ing 5 cycles, by which systems of 4 letters can neither be permuted nor left fixed. It follows that if H is contained in a doubly transitive group of degree $4p+9$, J is of degree 9 and doubly transitive. But no quotient group of the group of theorem V is simply isomorphic to a doubly transitive group of degree 9. An imprimitive group of degree $4p+10$ or $4p+14$ would be of class 10.* In the imprimitive group of degree $4p+12$ the systems of 4 letters each are permuted according to the alternating group of degree $p+3$. Since systems of two letters each are not possible, and the systems certainly contain 4 letters each, it is impossible for this group of degree $4p+12$ to have a subgroup of order p^2, even when $p=5$. Then its group J is transitive of degree 12 and of order $96m$, where m is greater than unity. Since the order of J is greater than $4!$ it contains an invariant operator which is regular, of degree 12, and is transformed into itself by the transitive subgroup of degree 8, so that this operator can only be of order 2 or 4, making J of order 96 at most. But we have seen that the order of J is greater than 96. Let H be of degree $4p+4$. Systems of 4 letters can be chosen in one way only. If H does not lead to a doubly transitive group of degree $4p+5$, it must lead to an imprimitive group of degree $4p+6$ with systems of two letters, which is in consequence of class 6 or less. If p is greater than 5, the doubly transitive group of degree $4p+5$ which we are led to consider can have no subgroup of order p^2, and hence J would necessarily be a doubly transitive group of degree 5, and that is impossible. Let $p=5$. Now $4p+5=25$, so that the order of G is a multiple of 125. Since P is a Sylow subgroup of the subgroup G_1 of G which leaves one letter fixed, P is invariant in a subgroup I whose constituent J is doubly transitive of order $20m$. Since the subgroup of order 5 in J is invariant, J is metacyclic. Such a group G^{25} is known to exist. It is the holomorph of the group of order 25 and type $(1,1)$. Its order is $25\cdot 24\cdot 20=12000$. However G^{25} is not a subgroup of a triply transitive group of degree 26, since our constituent J in such a group cannot be triply transitive of order $6\cdot 5\cdot 4=120$.

Let H be doubly transitive. Its degree is $4p+\lambda$, where we may assume that λ is greater than 5. The subgroup H_1 of H that leaves one letter fixed is imprimitive, for otherwise the substitutions of order p and degree $4p$ in it would generate a transitive group of degree $4p+\lambda-1$. H_1 has two systems of imprimitivity. The order of H_1 (when p is greater than 5) is not divisible by p^2, for then the substitutions of order p and of degree $5p$ in H_1 are incompatible with the systems of imprimitivity. If $p=5$, and 25 divides the order of H_1, each constituent of the group generated by the substitutions in H_1 of order p and degree $4p$ is of degree 15. The presence in a constituent of degree 15 of substitutions of degree 10 and also of degree 15, both of order 5, requires that it be primitive and alternating, thereby lowering the class of H to 6. Hence

*American Journal of Mathematics, vol. 28 (1904), p. 226.

the order of H_1 is not divisible by p^2. Then J, of degree λ is doubly transitive and hence has no invariant operators. But no subgroup of the octaedral group is doubly transitive on more than 4 letters, so that H cannot be doubly transitive.

It may be that H is a simply transitive primitive group. If H is of degree less than $4p+5$ it may lead to a doubly transitive group of degree $4p+5$, if our theorem is not true; but that, except for $p=5$, has been seen to be impossible. We now take up the various groups H in order.

H is of degree $4p+5$. The cyclic group P is now a Sylow subgroup of H_1, so that J is transitive of degree 5. Since H_1 is intransitive and has either two or three constituents, J is of order 10. This is certainly not possible unless $p=5$. Our theorem is still true unless H now leads to a doubly transitive group of degree $4p+6$ in which the group J (here of degree 6) is doubly transitive. But the substitutions of order p in this last group must generate an abelian group, and this is not true for a doubly transitive group of degree 6.

H is of degree $4p+6$. The order of H is not divisible by p^2 unless perhaps when $p=5$. But a substitution of order 5 and of degree 25 in H_1 requires that one transitive constituent of H_1 (of degree $5r+s$) have substitutions of order 5 both of degree $5r$ and $5(r-1)$ and hence be alternating. Then the class of H would not exceed 6. It follows that P is a Sylow subgroup of H_1 and J is a transitive group of degree 6. The transitive constituents of H_1 correspond to the 4 partitions of $4p+5$, thus:

$$3p+3, p+2;\ 2p+4, 2p+1;\ 2p+4, p+1, p;\ 2p+2, p+2, p+1.$$

The first partition requires that J be the symmetric group of degree 6, as is clearly impossible.

Now the non-regular transitive groups of degree 6 in which the quotient group with respect to an invariant substitution is a subgroup of the octaedral group may be readily set up. If there is an invariant substitution of order 3, there is one group:

(*a*) G^6_{18}, in which the quotient group is the non-abelian group of order 6.

If there is an invariant substitution of order-2 there are three groups:

(*b*) G^6_{48}, of class 2, with an octaedral quotient group. The subgroup leaving one letter fixed is octic.

(*c*) G^6_{24}, a subgroup of (*b*) corresponding to the tetraedral subgroup of its quotient group. The subgroup that leaves one letter fixed is the non-regular axial group.

(*d*) G^6_{12}, again a subgroup of (*b*), of class 4, in which the quotient group is of order 6.

If the identity is the invariant operator we have three groups:

(*e*) the octaedral group on 6 letters, written with respect to the cyclic group of order 4,

(f) with respect to an axial subgroup, and

(g) the tetraedral group put on 6 letters.

The last three groups are of class 4.

According to the second and third arrangements of the letters of H_1 in transitive sets, the constituent of degree $2p+4$ must be imprimitive, since it cannot be an alternating group. The group J belonging to it cannot be the octic group. This consideration bars these two partitions at once, since then J must contain a transposition. In the case of the last partition $(2p+2, p+2, p+1)$ there is in J_1 a substitution of order 2 and degree 4. Since this substitution is not invariant in J, it corresponds to operators in the tail of I which transpose systems. But only two systems may be transposed. Hence the entire operator of order 2 is a negative substitution, while H is a positive group.

H is of degree $4p+7$. A substitution of order 7 in J cannot be invariant and a subgroup of order 7 can be invariant only if $p=7$ and J is of class 6. There are only two possible partitions of $4p+6$ corresponding to the arrangements of the letters of H_1 in transitive sets which may permit J to be of class 6. One is $(3p+6, p)$ and since $\{A, B\}$ must have a constituent of degree $2p+2$, J is not of class 6. The other partition is $(2p+2, p+2, p+2)$. The group $\{A, B\}$ in this case has constituents of degrees $2p+1, p+1, p+1$. The constituent of degree $p+2=9$ in H_1 is triply transitive. The larger constituent is doubly transitive because of $\{A, B\}$ and hence the group of degree 9 is alternating. But a doubly transitive group of degree 16 may not have the alternating group of degree 9 as a quotient group. Or it is easy to see that the order of such a group may not be divisible by 81.

H is of degree $4p+8$. There are two possible partitions of $4p+7$, the degree of H_1: $3p+6, p+1$; and $2p+4, p+1, p+2$. If the order of H is a multiple of p^2, $p=5$ or 7, and hence a constituent of H_1 of degree $3p+6$ or $2p+4$ may not be imprimitive as none the less it must be. In the first place J is of class 2 or 3 and therefore not primitive. But an imprimitive group of degree 8 cannot have a substitution of degree and order 3 unless there are two systems of imprimitivity of 4 letters each. Two such systems however cannot be permuted by the transitive subgroup of degree 6. In case of the second partition the constituent of degree $2p+4$ must be imprimitive, and, since $\{A, B\}$ has a transitive constituent of degree $2p$ or $2p+2$, this constituent of degree $2p+4$ has a transitive subgroup of degree $2p+2$. Then in H the order of J is divisible by 64. But since the order of the quotient group of J taken with respect to the invariant substitution is a divisor of 24, the invariant operator is of order 8, an impossibility.

H is of degree $4p+9$. For the degree of H_1, the partition $(2p+4, 2p+4)$ is impossible inasmuch as each constituent is necessarily imprimitive, so that $\{A, B\}$ has one constituent of degree $2p+2$ and hence in $\{A, B, C\}$,

which is included in H_1, the substitution C may (by theorem III) be supposed to have been chosen in such a way that the last named group has one constituent of degree $3p + k$. Then the constituents of H_1 are of degrees $3p + 6$ and $p + 2$. The factor p^2 cannot enter the order of H. The order of J is a multiple of 27, but J cannot have an invariant operator of order 9.

If H is of degree $4p + 10$, $4p + 11$, or $4p + 12$, one constituent of H_1 certainly includes an alternating group, and in consequence H has substitutions of order p on less than 4 cycles.

We have now considered all possible cases that might lead to an exception to the theorem. It has been proved too that when $p = 5$, the degree of G may be as high as 25 but may not exceed 25 unless G includes the alternating group.

STANFORD UNIVERSITY,
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EXISTENCE AND OSCILLATION THEOREM FOR A CERTAIN BOUNDARY VALUE PROBLEM*

BY

GEORGE D. BIRKHOFF

It is the purpose of this paper to consider the existence and oscillation of the real solutions of a linear differential equation of the second order

$$\frac{d^2u}{dx^2} + q(x, \lambda)u = 0 \qquad (a \leqq x \leqq b), \tag{1}$$

subject to the self-adjoint boundary conditions

$$\begin{aligned} \alpha_0 u'(a) + \beta_0 u(a) &= \gamma_0 u'(b) + \delta_0 u(b), \\ \alpha_1 u'(a) + \beta_1 u(a) &= \gamma_1 u'(b) + \delta_1 u(b), \\ \alpha_0\beta_1 - \beta_0\alpha_1 &= \gamma_0\delta_1 - \delta_0\gamma_1, \end{aligned} \tag{2}$$

where the real coefficients α_0, β_0, γ_0, δ_0 and α_1, β_1, γ_1, δ_1 are not proportional.

The function $q(x, \lambda)$ is assumed to be continuous in (x, λ) for all real values of λ when x lies in (a, b) and to increase steadily with λ in such a way that

$$\lim_{\lambda=-\infty} q(x, \lambda) = -\infty, \qquad \lim_{\lambda=+\infty} q(x, \lambda) = +\infty. \tag{3}$$

We lay aside the trivial solution $u \equiv 0$.

STURM considered the problem under the special boundary conditions

$$\alpha_0 u'(a) + \beta_0 u(a) = 0, \qquad \gamma_1 u'(b) + \delta_1 u(b) = 0.\dagger \tag{4}$$

MASON‡ has proved the existence of an infinite number of values of λ furnishing a solution, when $q(x, \lambda)$ has the form $\lambda A(x) - g(x)$, and has given also an oscillation theorem for the special conditions

$$u(a) = u(b), \qquad u'(a) = u'(b). \tag{5}$$

BÔCHER§ employing other methods has removed MASON's restriction on

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† Liouville's Journal, ser. 1, vol. 1 (1836), pp. 106–186. His linear differential equation appears in a more general form.

‡ These Transactions, vol. 7 (1906), pp. 337–360.

§ Comptes Rendus, vol. 140 (1905), p. 928.

$q(x, \lambda)$ for the conditions (5) except that a certain uniqueness of the values λ is not proved. The method of this paper is like BÔCHER's, being based on certain theorems due to STURM.

§ 1. *Condition for a solution.*

In order to have a convenient notation let us write

$$(6)\quad \begin{aligned} L_0[u(x)] &\equiv \alpha_0 u'(x) + \beta_0 u(x), & M_0[u(x)] &\equiv \gamma_0 u'(x) + \delta_0 u(x), \\ L_1[u(x)] &\equiv \alpha_1 u'(x) + \beta_1 u(x), & M_1[u(x)] &\equiv \gamma_1 u'(x) + \delta_1 u(x). \end{aligned}$$

Furthermore we assume

$$(7)\quad \alpha_0\beta_1 - \beta_0\alpha_1 = \gamma_0\delta_1 - \delta_0\gamma_1 = 1,$$

as we may, since we can change either α_0, β_0, γ_0, δ_0, or α_1, β_1, γ_1, δ_1 by a constant factor.*

We now obtain an explicit condition that l shall be a value of λ for which there is a solution of (1), (2). Determine $u_0(x, \lambda)$, $u_1(x, \lambda)$ as the solutions of (1) for which

$$(8)\quad \begin{aligned} L_0[u_0(a, \lambda)] &= 0, & L_1[u_0(a, \lambda)] &= 1, \\ L_0[u_1(a, \lambda)] &= 1, & L_1[u_1(a, \lambda)] &= 0. \end{aligned}$$

These equations fix u_0, u_0', u_1, u_1' at $x = a$, and hence determine u_0 and u_1. We readily find that

$$(9)\quad L_0[u_0(x, \lambda)]L_1[u_1(x, \lambda)] - L_0[u_1(x, \lambda)]L_1[u_0(x, \lambda)] = -1,$$

for the left-hand member reduces to $u_0'u_1 - u_1'u_0 =$ constant, and this constant must be -1 by (8). Also we have

$$(10)\quad M_0[u_0(x, \lambda)]M_1[u_1(x, \lambda)] - M_0[u_1(x, \lambda)]M_1[u_0(x, \lambda)] = -1,$$

since here also the left-hand member reduces to $u_0'u_1 - u_1'u_0$.

The necessary and sufficient condition that there exists a solution $u \not\equiv 0$ when $\lambda = l$ is that $\phi(l) = 0$, where

$$(11)\quad \phi(\lambda) \equiv M_0[u_1(b, \lambda)] + M_1[u_0(b, \lambda)] - 2.$$

In fact the solutions u_0 and u_1 of (1) are linearly independent by (8), and therefore we may write any solution u of (1) in the form

$$c_0u_0 + c_1u_1.$$

If this expression be substituted for u in the conditions (2), we obtain, after sim-

* If, however,

$$\alpha_0\beta_1 - \beta_0\alpha_1 = \gamma_0\delta_1 - \delta_0\gamma_1 = 0,$$

this is not possible. In this case the conditions are of the Sturmian type (4) and we do not need to treat this case.

plifying by means of (8),

$$(12)\qquad \begin{aligned} c_1 &= c_0 M_0[u_0(b,\lambda)] + c_1 M_0[u_1(b,\lambda)], \\ c_0 &= c_0 M_1[u_0(b,\lambda)] + c_1 M_1[u_1(b,\lambda)]. \end{aligned}$$

These two equations are linear in c_0 and c_1; a solution $u \not\equiv 0$ which satisfies the conditions (1) and (2) will therefore exist if and only if the determinant

$$(13)\qquad \phi(\lambda) \equiv \begin{vmatrix} -M_0[u_0(b,\lambda)], & 1 - M_0[u_1(b,\lambda)] \\ 1 - M_1[u_0(b,\lambda)], & -M_1[u_1(b,\lambda)] \end{vmatrix}$$

vanishes. By means of (10) for $x = b$, this identity reduces to (11).

§ 2. *Simple and double solutions.*

A value l of λ for which a solution $u \not\equiv 0$ of (1), (2) exists is said to be *simple* if all the solutions are linearly dependent: if, however, there exist two linearly independent solutions of (1), (2) for $\lambda = l$, the value l is said to be *double*. *The necessary and sufficient condition that $\lambda = l$ is a double value is that*

$$(14)\quad M_0[u_0(b,l)] = M_1[u_1(b,l)] = 0, \qquad M_0[u_1(b,l)] = M_1[u_0(b,l)] = 1.$$

In fact two linearly independent solutions of (1), (2) exist if and only if the elements of the determinant $\phi(\lambda)$ given by (13) all vanish.

The question now arises: How does $\phi(\lambda)$ behave at simple and double values? It is the object of this section to show that $\phi(\lambda)$ changes sign at a simple value of λ and preserves its sign at a double value.

It is essential first to derive some formulas. Let us change λ to $\bar{\lambda} = \lambda + \delta\lambda$; all functions f of λ will then change to $\bar{f} = f + \delta f$. The function $\delta u_0(x, \lambda)$ fulfills the conditions

$$(15)\qquad \frac{d^2}{dx^2}[\delta u_0(x,\lambda)] + q(x,\lambda)\delta u_0(x,\lambda) = -\delta q(x,\lambda)\bar{u}_0(x,\lambda),$$

$$(16)\qquad \delta u_0(a,\lambda) = \frac{d}{dx}[\delta u_0(a,\lambda)] = 0;$$

the equation (15) being obtained by subtracting from one another the equations satisfied by u_0 and $\bar{u}_0$, while the equations (16) are a consequence of the fact that $u_0(a,\lambda)$ and $u_0'(a,\lambda)$, as determined from (8), are independent of λ. A like set of equations holds for $\delta u_1(x,\lambda)$.

From the non-homogeneous linear differential equation (15) and the conditions (16), the function $\delta u_0(x,\lambda)$ can be explicitly obtained; we have

$$(17)\quad \delta u_0(x,\lambda) = \int_a^x [u_0(x,\lambda)u_1(\xi,\lambda) - u_1(x,\lambda)u_0(\xi,\lambda)]\delta q(\xi,\lambda)\bar{u}_0(\xi,\lambda)\,d\xi,$$

as may be proved by a direct substitution. If we differentiate (17) as to x we obtain

$$(18)\quad \frac{d}{dx}[\delta u_0(x,\lambda)] = \int_a^x [u_0'(x,\lambda)u_1(\xi,\lambda) - u_1'(x,\lambda)u_0(\xi,\lambda)]\,\delta q(\xi,\lambda)\bar{u}_0(\xi,\lambda)d\xi.$$

From (17) and (18) we get

$$(19)\quad \delta M_0[u_0(b,\lambda)] = \int_a^b \{M_0[u_0(b,\lambda)]u_1(\xi,\lambda) - M_0[u_1(b,\lambda)]u_0(\xi,\lambda)\}\,\delta q(\xi,\lambda) \times \bar{u}_0(\xi,\lambda)d\xi,$$

$$(20)\quad \delta M_1[u_0(b,\lambda)] = \int_a^b \{M_1[u_0(b,\lambda)]u_1(\xi,\lambda) - M_1[u_1(b,\lambda)]u_0(\xi,\lambda)\}\,\delta q(\xi,\lambda) \times \bar{u}_0(\xi,\lambda)d\xi,$$

and in like manner one can prove

$$(21)\quad \delta M_0[u_1(b,\lambda)] = \int_a^b \{M_0[u_0(b,\lambda)]u_1(\xi,\lambda) - M_0[u_1(b,\lambda)]u_0(\xi,\lambda)\}\,\delta q(\xi,\lambda) \times \bar{u}_1(\xi,\lambda)d\xi,$$

$$(22)\quad \delta M_1[u_1(b,\lambda)] = \int_a^b \{M_1[u_0(b,\lambda)]u_1(\xi,\lambda) - M_1[u_1(b,\lambda)]u_0(\xi,\lambda)\}\,\delta q(\xi,\lambda) \times \bar{u}_1(\xi,\lambda)d\xi.$$

We are now in a position to consider the sign of ϕ near a value l of λ for which $\phi = 0$. We first take the case when l is a simple value.

By means of (11) we have

$$\delta\phi = \delta M_0[u_1(b,\lambda)] + \delta M_1[u_0(b,\lambda)].$$

Therefore by (20) and (21) we obtain

$$(23)\quad \delta\phi = \int_a^b \{M_0[u_0(b,\lambda)]u_1(\xi,\lambda)\bar{u}_1(\xi,\lambda) - M_0[u_1(b,\lambda)]u_0(\xi,\lambda)\bar{u}_1(\xi,\lambda) + M_1[u_0(b,\lambda)]u_1(\xi,\lambda)\bar{u}_0(\xi,\lambda) - M_1[u_1(b,\lambda)]u_0(\xi,\lambda)\bar{u}_0(\xi,\lambda)\}\,\delta q(\xi,\lambda)d\xi.$$

It is our purpose to investigate the sign of $\delta\phi$ for small $\delta\lambda$ by the aid of this formula.

The integrand on the right is the product of two factors, the second of which, $\delta q(\xi,\lambda)$, has the sign of $\delta\lambda$, since $q(\xi,\lambda)$ by hypothesis increases with λ. As the quantity $\delta\lambda$ becomes smaller, $\bar{u}_0$ and $\bar{u}_1$ approach u_0 and u_1, and the first factor of the integrand approaches a homogeneous quadratic form in $u_0(\xi,\lambda)$ and $u_1(\xi,\lambda)$ which is obtained when we replace $\bar{u}_0(\xi,\lambda)$ and $\bar{u}_1(\xi,\lambda)$ by $u_0(\xi,\lambda)$ and $u_1(\xi,\lambda)$ respectively. The discriminant of this quadratic form is

$$\{-M_0[u_1(b,\lambda)] + M_1[u_0(b,\lambda)]\}^2 + 4M_0[u_0(b,\lambda)]M_1[u_1(b,\lambda)].$$

If here we substitute the value

$$M_0[u_0(b,\lambda)]M_1[u_1(b,\lambda)] = -1 + M_0[u_1(b,\lambda)]M_1[u_0(b,\lambda)]$$

obtained from (10), the discriminant becomes

$$\{M_0[u_1(b,\lambda)] + M_1[u_0(b,\lambda)]\}^2 - 4,$$

which vanishes at $\lambda = l$ by (11) since $\phi(l) = 0$.

Now it is clear that the three coefficients of the form are not zero except at a double value; otherwise with the aid of (10) we derive equations (14). Also the functions u_0 and u_1 are linearly independent. Thus as $\bar{\lambda}$ tends to a value l, the limit of the first factor of the integrand is an expression of definite sign for $a \leqq \xi \leqq b$, the same as that of $M_0[u_0(b,l)]$ and $M_1[u_1(b,l)]$, save at isolated points at which it vanishes.

We conclude that *at a simple value l of λ, $\phi(\lambda)$ changes sign in such a way that $\delta\phi/\delta\lambda$ is of the same sign as $M_0[u_0(b,\lambda)]$ and $M_1[u_1(b,\lambda)]$.**

Let us now pass to the case of a double value l. If we recall conditions (14), we find that for $\lambda = l$

$$M_0[u_0(b,\bar{\lambda})] = \delta M_0[u_0(b,\lambda)], \qquad M_0[u_1(b,\bar{\lambda})] = \delta M_0[u_1(b,\lambda)],$$
$$M_0[u_1(b,\bar{\lambda})] = 1 + \delta M_0[u_1(b,\lambda)], \quad M_1[u_1(b,\bar{\lambda})] = 1 + \delta M_1[u_1(b,\lambda)].$$

Substitute these values in (10), taking $\lambda = \bar{\lambda}$, and we find

$$\delta M_0[u_0(b,\lambda)]\,\delta M_1[u_1(b,\lambda)] - \{\delta M_0[u_1(b,\lambda)] + \delta M_1[u_0(b,\lambda)] + \delta M_0[u_1(b,\lambda)]\,\delta M_1[u_0(b,\lambda)]\} = 0$$

so that

$$\delta\phi = \delta M_0[u_0(b,\lambda)]\,\delta M_1[u_1(b,\lambda)] - \delta M_0[u_1(b,\lambda)]\,\delta M_1[u_0(b,\lambda)].$$

* This argument is satisfactory when the λ-derivative of $q(x,\lambda)$ exists on either side of l and is positive for some x of (a,b).

Assume now that the above statement is not true for ϕ on one side of l, say for $\lambda > l$. Replace q in the interval $(l, l+d)$ by

$$q^*(x,\lambda) = q(x,l) + \frac{\lambda - l}{d}\,q(x, l+d),$$

which satisfies the above condition. Then we have

$$\phi^*(l) = \phi(l) = 0, \qquad \phi^*(l+d) = \phi(l+d).$$

But the above statement does hold for ϕ^*. From this it follows that for a proper choice of d, as small as we please, we shall have $\phi^*(l') = 0$ where $l < l' \leqq l + d$. But for two successive simple values for which $\phi^*(\lambda) = 0$, $M_0[u_0^*(b,\lambda)]$ and $M_1[u_1^*(b,\lambda)]$ must change sign (or vanish) since the sign of $\delta\phi^*/\delta\lambda$ does, and at a double value both vanish. We conclude that

$$M_0[u_0^*(b,\lambda_1)] = 0, \qquad M_1[u_1^*(b,\lambda_2)] = 0, \qquad (l < \lambda_1, \lambda_2 \leqq l + d).$$

Let d now tend to zero. Then we infer

$$M_0[u_0(b,l)] = M_1[u_1(b,l)] = 0.$$

Since $\phi(l) = 0$ these relations, combined with (10) when $x = l$, show that l would then be a double value, contrary to hypothesis.

If now the expressions (19), (20), (21), (22) be simplified by the use of (14) and substituted in this expression for $\delta\phi$, and if we write for $\bar{u}_0(\xi, \lambda)$, $\bar{u}_1(\xi, \lambda)$ their limiting values $u_0(\xi, \lambda)$, $u_1(\xi, \lambda)$, we obtain

$$(24) \qquad -\int_a^b u_0^2(\xi, \lambda)\delta q(\xi, \lambda)d\xi \int_a^b u_1^2(\xi, \lambda)\delta q(\xi, \lambda)d\xi + \left[\int_a^b u_0(\xi, \lambda)u_1(\xi, \lambda)\delta q(\xi, \lambda)d\xi\right]^2.$$

But by a familiar inequality

$$\int_a^b f^2 d\xi \int_a^b g^2 d\xi - \left[\int_a^b fg d\xi\right]^2 > 0,$$

provided f and g are linearly independent real continuous functions. Hence, if we write

$$f = u_0(\xi, \lambda)\sqrt{\pm \delta q(\xi, \lambda)}, \qquad g = u_1(\xi, \lambda)\sqrt{\pm \delta q(\xi, \lambda)},$$

using + or − according as $\delta\lambda$ is positive or negative, it is apparent that the quantity (24) is negative. Therefore $\phi(\lambda)$ *preserves a negative sign at a double value of* λ.*

§ 3. *The existence theorem.*

Now let the infinite set of values of λ furnishing a solution u of (1) for the Sturmian conditions

$$(25) \qquad L_0[u(a)] = 0, \qquad M_0[u(b)] = 0,$$

be denoted by

$$\lambda_1, \lambda_2, \cdots \qquad (\lambda_1 < \lambda_2 < \cdots).$$

These values separate the λ-axis into the intervals

$$(26) \qquad (-\infty, \lambda_1), \quad (\lambda_1, \lambda_2), \quad (\lambda_2, \lambda_3), \cdots.$$

This division into intervals is not uniquely determined, for the conditions (2) can be replaced by any two linearly independent conditions which arise from (2) by linear combination.

THE EXISTENCE THEOREM. *There exists an infinite set of values* $l_1, l_2, \cdots$ *of* λ *furnishing solutions of* (1), (2). *If we take these quantities in order of increasing magnitude counting each double value twice, there are the following*

* This argument is not satisfactory unless the λ-derivative of $q(x, \lambda)$ exists on either side of l, and is positive for $a \leqq x \leqq b$.

If the derivative does not exist, we proceed as in the previous footnote and prove that $M_0[u_0^*(b, \lambda)]$ and $M_1[u_1^*(b, \lambda)]$ vanish in the vicinity of l as well as at l. This is impossible since the zeros of these functions are separated by finite intervals according to the results of STURM.

possible cases:

$$\mathrm{I}_a \qquad \lambda_1 < l_1 \leqq \lambda_2 \leqq l_2 < \lambda_3 < l_3 \leqq \lambda_4 \leqq l_4 < \lambda_5 \cdots,$$

$$\mathrm{I}_b \qquad l_1 < \lambda_1 < l_2 \leqq \lambda_2 \leqq l_3 < \lambda_3 < l_4 \leqq \lambda_4 \leqq l_5 < \lambda_5 \cdots,$$

$$\mathrm{II}_a \qquad \lambda_1 \leqq l_1 < \lambda_2 < l_2 \leqq \lambda_3 \leqq l_3 < \lambda_4 < l_4 \leqq \lambda_5 \cdots,$$

$$\mathrm{II}_b \qquad l_1 \leqq \lambda_1 \leqq l_2 < \lambda_2 < l_3 \leqq \lambda_3 \leqq l_4 < \lambda_4 < l_5 \leqq \lambda_5 \cdots.$$

Proof. Clearly u_0 is the only solution of (1) (save for a constant factor) which satisfies the first condition (25), and if the second of these conditions is also to be fulfilled, we must have $M_0[u_0(b, \lambda)] = 0$. Hence we see that

$$M_0[u_0(b, \lambda)] = 0 \qquad \text{for } \lambda = \lambda_1, \lambda_2, \cdots, \tag{27}$$

but for no other λ. It follows from (10) that

$$M_0[u_1(b, \lambda_i)]M_1[u_0(b, \lambda_i)] = 1 \qquad (i = 1, 2, \cdots). \tag{28}$$

Substituting the value for $M_0[u_1(b, \lambda_i)]$ obtained from this last equation in (11), we find

$$\phi(\lambda_i) = \frac{1}{M_1[u_0(b, \lambda_i)]}\{1 - M_1[u_0(b, \lambda_i)]\}^2 \qquad (i = 1, 2, \cdots). \tag{29}$$

We infer that ϕ *has the same sign as* $M_1[u_0(b, \lambda)]$ *at the values* $\lambda = \lambda_1, \lambda_2, \cdots,$ *unless* $M_1[u_0(b, \lambda)] = 1$, *when* ϕ *vanishes.*

But since $u_0(a, \lambda)$, $u_0'(a, \lambda)$ are independent of λ, it is a consequence of familiar theorems due to STURM that the roots of $M_0[u_0(b, \lambda)] = 0$ and $M_1[u_0(b, \lambda)] = 0$ separate each other, and that $M_0[u_0(b, \lambda)]$ and $M_1[u_0(b, \lambda)]$ change sign when they vanish.*

Accordingly by (27) it is clear that $M_1[u_0(b, \lambda)]$ *alternates in sign at the values* $\lambda_1, \lambda_2 \cdots$. Either $M_1[u_0(b, \lambda)]$ is positive at $\lambda_1, \lambda_3, \cdots$ and negative at $\lambda_2, \lambda_4, \cdots$, or *vice versa.* Thus two cases arise:

$$\text{(30)} \qquad \begin{cases} \text{CASE I} & \begin{cases} \phi \geqq 0 \text{ at } \lambda_1, \lambda_3, \cdots, \\ \phi < 0 \text{ at } \lambda_2, \lambda_4, \cdots, \\ \qquad \text{when } M_1[u_0(b, \lambda_1)] > 0. \end{cases} \\ \text{CASE II} & \begin{cases} \phi < 0 \text{ at } \lambda_1, \lambda_3, \cdots, \\ \phi \geqq 0 \text{ at } \lambda_2, \lambda_4, \cdots, \\ \qquad \text{when } M_1[u_0(b, \lambda_1)] < 0. \end{cases} \end{cases}$$

We see that there must exist values l of λ as follows. In Case I there exist at least two values l, say l_1, l_2, in each double interval $(\lambda_{2p}, \lambda_{2p+2})$,

* Loc. cit., p. 139, 142.

$p = 1, 2, \cdots$, such that

$$\lambda_{2p} < l_1 \leqq \lambda_{2p+1} \leqq l_2 < \lambda_{2p+2},$$

and at least one l_2 such that

$$\lambda_1 \leqq l_2 < \lambda_2.$$

In Case II there exist at least two values l_1, l_2 in each double interval $(\lambda_{2p-1}, \lambda_{2p+1})$, $p = 1, 2 \cdots$ such that

$$\lambda_{2p-1} < l_1 \leqq \lambda_{2p} \leqq l_2 < \lambda_{2p+1}.$$

For this deduction it is necessary to know (see § 2) that ϕ preserves its sign only at a double value of l which we count as two.

It remains to prove first that there exist only the two values l_1, l_2 in the double intervals; secondly that there exist in Case I either two values l_1, l_2 in $(-\infty, \lambda_2)$ such that

$$l_1 \leqq \lambda_1 \leqq l_2 < \lambda_2 \qquad \text{(Subcase } \mathrm{I}_b\text{)},$$

or one value l_2 such that

$$\lambda_1 \leqq l_2 < \lambda_2 \qquad \text{(Subcase } \mathrm{I}_a\text{)};$$

and lastly that there exists in Case II only one value l_2 in $(-\infty, \lambda_1)$ such that

$$l_2 \leqq \lambda_1. \qquad \text{(Subcase } \mathrm{II}_b\text{)},$$

or none (Subcase II_a).

Let us take up the first point. If there exist additional values l in some double interval $(\lambda_q, \lambda_{q+2})$, there will be at least four, since $\phi(\lambda_q)$ and $\phi(\lambda_{q+2})$ have the same sign. If there is no double value, there must at least two simple values fall within one of the intervals $(\lambda_q, \lambda_{q+1})$ or $(\lambda_{q+1}, \lambda_{q+2})$, at which $\delta\phi/\delta\lambda$ is of opposite signs. But this is impossible since by the last section $\delta\phi/\delta\lambda$ has at all such values within one interval the sign of $M_0[u_0(b, \lambda)]$, which is invariable within the interval. On the other hand if a double value exists, it must fall at λ_{q+1} by (14); then ϕ in the neighborhood of λ_{q+1} is negative by § 2. Other values l besides the double value would then imply at least two roots l existing in $(\lambda_q, \lambda_{q+1})$ or $(\lambda_{q+1}, \lambda_{q+2})$, inasmuch as $\phi(\lambda_q)$ and $\phi(\lambda_{q+2})$ are negative. This is impossible, as we have just seen.

Thus the first point is proved, and the second and third may be treated in the same manner.

§ 4. *Discrimination between the four sub-cases.*

We have Case I or Case II according as $M_1[u_0(b, \lambda_1)] > 0$ or $M_1[u_0(b, \lambda_1)] < 0$, and under each of these subcase a or b according as ϕ is of the same sign as $M_1[u_0(b, \lambda_1)]$ or of opposite sign, for large negative λ. This is evident from (30).

It remains now to determine the sign of these quantities.

Sign of $M_1[u_0(b, \lambda_1)]$.

We have $M_0[u_0(b, \lambda)] = 0$ for $\lambda = \lambda_1$ but not for $\lambda < \lambda_1$. By theorems of STURM * $u_0(x, \lambda)$ does not vanish for $a < x < b$ and $\lambda \leqq \lambda_1$. There are four sub-cases:

(1) $\alpha_0 \neq 0$, $\gamma_0 \neq 0$.

Since $\gamma_0 u_0'(b, \lambda_1) + \delta_0 u_0(b, \lambda_1) = 0$, we have by (7)

$$M_1[u_0(b, \lambda_1)] = \gamma_1\left(-\frac{\delta_0}{\gamma_0}u_0(b, \lambda_1)\right) + \delta_1 u_0(b, \lambda_1) = \frac{u_0(b, \lambda_1)}{\gamma_0}.$$

But $u_0(b, \lambda_1)$ has the same sign as $u_0(a, \lambda_1) = \alpha_0$. Hence the *sign of* $M_1[u_0(b, \lambda_1)]$ *is that of* α_0/γ_0.

(2) $\alpha_0 \neq 0$, $\gamma_0 = 0$.

Here we have $u_0(b, \lambda_1) = 0$, and $u_0'(b, \lambda_1)$ will have the opposite sign to $u_0(a, \lambda_1) = \alpha_0$. Therefore the *sign of* $M_1[u_0(b, \lambda_1)]$ *is that of* $-\alpha_0\gamma_1$.

(3) $\alpha_0 = 0$, $\gamma_0 \neq 0$.

Here, as in case (1),

$$M_1[u_0(b, \lambda_1)] = \frac{u_0(b, \lambda_1)}{\gamma_0}.$$

But $u_0(b, \lambda_1)$ has the same sign as $u_0'(a, \lambda_1) = -\beta_0$ since $u_0(a, \lambda_1) = \alpha_0 = 0$. *The sign of* $M_1[u_0(b, \lambda_1)]$ *is that of* $-\beta_0/\gamma_0$.

(4) $\alpha_0 = 0$, $\gamma_0 = 0$.

Here, as in case (2),

$$M_1[u_0(b, \lambda_1)] = \gamma_1 u_0'(b, \lambda_1),$$

where $u_0'(b, \lambda_1)$ has the sign opposite to $u_0'(a, \lambda_1) = -\beta_0$ since $u_0(a, \lambda_1) = 0$ and $u_0(b, \lambda_1) = 0$. *The sign of* $M_1[u_0(b, \lambda_1)]$ *is that of* $-\beta_0\delta_0$.

Sign of $\phi(\lambda)$ *for large negative* λ.

It remains to determine the sign of $\phi(\lambda)$ for large negative λ. Let us assume first that $\alpha_0 \neq 0$.

We have, in view of the relation (10) between the solutions of (1), the equation

$$u_1(x, \lambda) = \left(m + n\int_a^x \frac{dx}{[u_0(x, \lambda)]^2}\right)u_0(x, \lambda).$$

By means of (8) we determine the values of the constants m and n and

* Loc. cit., p. 140.

have

$$u_1(x,\lambda)=\left(-\frac{\alpha_1}{\alpha_0}+\int_a^x\frac{dx}{[u_0(x,\lambda)]^2}\right)u_0(x,\lambda).$$

If we substitute this expression for $u_1(x,\lambda)$ in (11), we find

$$\begin{aligned}\phi(\lambda)&=\gamma_1u_0'(b,\lambda)+\delta_1u_0(b,\lambda)\\ &\quad+\gamma_0\left[\left(-\frac{\alpha_1}{\alpha_0}+\int_a^b\frac{dx}{[u_0(x,\lambda)]^2}\right)u_0'(b,\lambda)+\frac{1}{u_0(b,\lambda)}\right]\\ &\quad+\delta_0\left(-\frac{\alpha_1}{\alpha_0}+\int_a^b\frac{dx}{[u_0(x,\lambda)]^2}\right)u_0(b,\lambda)-2.\end{aligned}\tag{31}$$

Now let us recall that for $x>a$

$$\lim_{\lambda=-\infty}u_0'(x,\lambda)=\lim_{\lambda=-\infty}u_0(x,\lambda)=\infty,$$

$$\lim_{\lambda=-\infty}\frac{u_0'(x,\lambda)}{u_0(x,\lambda)}=+\infty.$$

It is then apparent that $\phi(\lambda)$ has the sign of $(\alpha_0\gamma_1-\gamma_0\alpha_1)u_0'(b,\lambda)/\alpha_0$ when λ is large, if $\alpha_0\gamma_1-\gamma_0\alpha_1\neq 0$. But $u_0'(b,\lambda)$ has the sign of $u_0(a,\lambda)=\alpha_0$. Hence *for large negative* λ *the sign of* $\phi(\lambda)$ *is that of* $(\alpha_0\gamma_1-\gamma_0\alpha_1)$, *provided that* $\alpha_0\gamma_1-\gamma_0\alpha_1\neq 0$.

If $\alpha_0=0$, the above statement remains true. In this case we express $u_0(x,\lambda)$ in terms of $u_1(x,\lambda)$ since $\alpha_1\neq 0$.

Let us next attend to the case where $\alpha_0\gamma_1-\gamma_0\alpha_1=0$ but $\alpha_0\neq 0$. In this case we transform the second of conditions (2) so that $\alpha_1=\gamma_1=0$ by linear combination. Furthermore we choose a multiplier so that (7) also holds, whence $\beta_1=1/\alpha_0$, $\delta_1=1/\gamma_0$.

When these values of α_1, β_1, γ_1, δ_1 are substituted in (31) we get

$$\begin{aligned}\phi(\lambda)&=\gamma_0\left(\frac{1}{\gamma_0^2}u_0(b,\lambda)+\int_a^b\frac{dx}{[u_0(x,\lambda)]^2}u_0'(b,\lambda)+\frac{1}{u_0(b,\lambda)}\right)\\ &\quad+\delta_0\int_a^b\frac{dx}{[u_0(x,\lambda)]^2}u_0(b,\lambda)-2.\end{aligned}\tag{32}$$

Now $u_0(b,\lambda)$, $u_0'(b,\lambda)$ have the same sign as $u_0(a,\lambda)=\alpha_0$. Therefore by (32), *for large negative* λ, $\phi(\lambda)$ *has the sign of* $\alpha_0\gamma_0$ *in the case* $\alpha_0\gamma_1-\gamma_0\alpha_1=0$, $\alpha_0\neq 0$, *if the second condition of* (2) *is so chosen that*

$$\alpha_1=0,\qquad \beta_1=\frac{1}{\alpha_0},\qquad \gamma_1=0,\qquad \delta_1=\frac{1}{\gamma_0}.$$

Finally let us consider the case where $\alpha_0\gamma_1-\gamma_0\alpha_1=0$ and $\alpha_0=0$. In this case α_1 is not zero since $\alpha_0\beta_1-\beta_0\alpha_1=1$. By symmetry it appears that *for large negative* λ, $\phi(\lambda)$ *has the sign of* $-\alpha_1\gamma_1$ *if both* $\alpha_0\gamma_1-\gamma_0\alpha_1=0$ *and* $\alpha_0=0$.

It should be noted that the above classification depends on the selection of the first condition $L_0[u(a)] = M_0[u(b)]$ out of the double infinity of conditions.

§ 5. *The oscillation theorem.*

THEOREM. *The solution $u_p(x)$ of* (1), (2) *which corresponds to $\lambda = l_p$ vanishes $p-1$, p, or $p+1$ times for $a < x \leqq b$ in accordance with the following table, in which*

$$\bar{\alpha}_0 = \gamma_0\alpha_1 - \alpha_0\gamma_1,\ \bar{\beta}_0 = \gamma_0\beta_1 - \beta_0\gamma_1, \quad \text{and} \quad K = -\bar{\beta}_0/\bar{\alpha}_0.*$$

I.	$\bar{\alpha}_0 > 0$,	$p = 2m$,	$\dfrac{u_p'(a)}{u_p(a)} \leqq K$;	$p+1$ times,
			$\dfrac{u_p'(a)}{u_p(a)} > K$;	p times,
		$p = 2m+1$,	$\dfrac{u_p'(a)}{u_p(a)} \leqq K$;	p times,
			$\dfrac{u_p'(a)}{u_p(a)} > K$;	$p-1$ times,
II.	$\bar{\alpha}_0 < 0$,	$p = 2m$,	$\dfrac{u_p'(a)}{u_p(a)} \leqq K$	p times,
			$\dfrac{u_p'(a)}{u_p(a)} > K$;	$p-1$ times,
		$p = 2m+1$,	$\dfrac{u_p'(a)}{u_p(a)} \leqq K$;	$p+1$ times,
			$\dfrac{u_p'(a)}{u_p(a)} > K$;	p times,
III.	$\bar{\alpha}_0 = 0$, $\bar{\beta}_0 > 0$,	$p = 2m$;		p times,
		$p = 2m+1$;		$p+1$ times,
	$\bar{\alpha}_0 = 0$, $\bar{\beta}_0 < 0$,	$p = 2m$;		$p+1$ times,
		$p = 2m+1$;		p times.

Proof. The conditions may always be written in the form †

$$\begin{aligned} &\bar{\alpha}_0 u'(a) + \bar{\beta}_0 u(a) = u(b), \\ &\bar{\alpha}_1 u'(a) + \bar{\beta}_1 u(a) = -u'(b) + \bar{\delta}_1 u(b). \end{aligned} \tag{33}$$

* At a double value we can take any solution $u_p(x)$ as corresponding to λ_p. Also if $u_p(a) = 0$, we make the convention that $u_p'(a)/u_p(a) = -\infty$.

† If the conditions be taken in this form at the start, no transformation is necessary. To obtain the first of these new conditions from (2), multiply the first by $-\gamma_0$, the second by γ_1, and add. The second of these conditions is then any condition so taken as to satisfy (7).

Then $\bar{\phi}(\lambda)$ has the sign of $-\bar{\alpha}_0\bar{\gamma}_1 = \bar{\alpha}_0$ at λ_1 and of $(\bar{\alpha}_0\bar{\gamma}_1 - \bar{\gamma}_0\bar{\alpha}_1) = \bar{\alpha}_0$ for large negative λ. This follows from the last section. With these new conditions we therefore have case I_a or II_a of the existence theorem. Accordingly by the existence theorem, the value l_p lies on the interval $(\lambda_p, \lambda_{p+1})$.

By well-known theorems of STURM, $\bar{u}_0(x, \lambda)$ will vanish p times on $a < x < b$ for $\lambda_p \leqq \lambda \leqq \lambda_{p+1}$. But the roots of $u_p(x)$ and $\bar{u}_0(x, l_p)$ separate each other, or else coincide. Hence $u_p(x)$ *vanishes* $p-1$, p, *or* $p+1$ *times on* $a < x \leqq b$.

Consider first the case $\bar{\alpha}_0 > 0$, and assume that $u_p(a)$ and $u_p(b)$ do not vanish, a retriction which is easily removed. If we have $p = 2m$ and $u_p'(a)/u_p(a) \leqq K$, we proceed as follows: Since, by definition the equation $\bar{u}_0'(a, l_p)/\bar{u}_0(a, l_p) = K$ holds, the function $u_p(x)$ vanishes at least as often as $\bar{u}_0(x, l_p)$ and hence p or $p+1$ times. But also from (33) we have $\bar{\alpha}_0 u_p(a)[u_p'(a)/u_p(a) - K] = u_p(b)$. Thus $u_p(a)$ and $u_p(b)$ have opposite signs, and $u_p(x)$ vanishes an odd number of times. This excludes the first possibility. Hence $u_p(x)$ vanishes $p+1$ times.

Likewise if $p = 2m$ and $u_p'(a)/u_p(a) > K$, we see that $u_p(x)$ cannot vanish more often than $\bar{u}_0(x, l_p)$, and therefore vanishes $p-1$ or p times. However, $u_p(a)$ and $u_p(b)$ have like signs, so that $u_p(x)$ must vanish p times.

Also if $p = 2m+1$, $u_p'(a)/u_p(a) \leqq K$, we find that $u_p(x)$ must vanish p times.

Also if $p = 2m+1$, $u_p'(a)/u_p(a) > K$, we find that $u_p(x)$ must vanish $p-1$ times.

A precisely similar discussion is possible when $\bar{\alpha}_0 < 0$.

When one has $\bar{\alpha}_0 = 0$, $\bar{\beta}_0 > 0$, the first condition (33) shows that $u_p(a)$ and $u_p(b)$ have like signs. Also $u_p(x)$ has at most p or $p-1$ zeros, since $\bar{u}_0(x, l_p)$ vanishes for $x = a$. From this follows the table for this case.

In a like manner one may discuss the case $\bar{\alpha}_0 = 0$, $\bar{\beta}_0 < 0$.

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ON THE REGIONS OF CONVERGENCE OF POWER-SERIES WHICH REPRESENT TWO-DIMENSIONAL HARMONIC FUNCTIONS*

BY

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The function $u(x, y)$ is said to be harmonic throughout a two-dimensional continuum T in the (x, y)-plane† if it is continuous, has continuous first and second partial derivatives, and satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

at every point of T. In this case it is well known that u is analytic throughout T, that is, it can be developed about every point of T into a double power-series. The main object of the present note is to determine the region of convergence of this double series.‡ For the sake of comparison I have also considered the simple series obtained from it by grouping together the terms of the same degree. In order to avoid a possible misunderstanding, I make the explicit statement that a double series shall be regarded as convergent when and only when every simple series into which its terms can be rearranged converges. According to this convention, every convergent double series is absolutely convergent.

The main results of this paper are recapitulated in § 3. Attention may also be called to the result stated at the close of § 1.

1. The double power-series. We approach our problem from the side of the Cauchy-Kowalewski existence theorem. Let $f_0(x)$ and $f_1(x)$ be two functions analytic when $x = 0$, and denote by K_0 and K_1 the radii of convergence

* Presented to the Society October 25, 1902, and subsequently further developed. Cf. Bulletin of the American Mathematical Society, series 2, vol. 9 (1903), p. 186-7, and OSGOOD, *Funktionentheorie*, vol. 1, p. 575.

† By a two-dimensional continuum I understand a connected region of the plane every point of which is an internal point. Throughout this note only real quantities are considered except where the contrary is explicitly stated.

‡ The only investigation of this question with which I am acquainted is contained in PICARD's *Traité d'Analyse* (1st or 2d edition), vol. 2, Chapter 1, § 14. It is there shown that if R is the distance from any point (x_0, y_0) of T to the nearest point of the boundary, the development about (x_0, y_0) converges throughout the square

$$|x - x_0| < \tfrac{1}{2}R, \qquad |y - y_0| < \tfrac{1}{2}R.$$

The present note shows that this is in reality only a portion of the region of convergence.

of the developments of these functions about the point $x = 0$. If we denote by K the smaller of these two quantities, we may write

$$f_0(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$f_1(x) = b_0 + b_1 x + b_2 x^2 + \cdots \qquad (|x| < K).$$

Let us now try to find in the neighborhood of the origin a solution of (1) which satisfies the auxiliary conditions

$$(2) \qquad u\big]_{y=0} = f_0(x), \qquad \frac{\partial u}{\partial y}\Big]_{y=0} = f_1(x).$$

Suppose that u is written as a double power-series in (x, y) with undetermined coefficients, and let us try to determine these coefficients in such a way that (1) and (2) are satisfied. If we arrange the double power-series according to powers of y, the first two terms will be uniquely determined by conditions (2), and we may write

$$(3) \qquad u(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots$$

where $f_2(x), f_3(x), \cdots$ are power-series in x with undetermined coefficients. Substituting this series in (1), we readily find the formulæ

$$f_{2n}(x) = \frac{(-1)^n}{(2n)!} f_0^{[2n]}(x)$$
$$(n = 1, 2, 3, \cdots).$$
$$f_{2n+1}(x) = \frac{(-1)^n}{(2n+1)!} f_1^{[2n]}(x)$$

We thus see that there is only one double power-series in (x, y) which even formally satisfies (1) and (2). This series we will call S. When rearranged according to powers of y, the series S becomes*

$$(4) \quad f_0(x) + f_1(x)y - \frac{f_0''(x)}{2!}y^2 - \frac{f_1''(x)}{3!}y^3 + \frac{f_0^{IV}(x)}{4!}y^4 + \frac{f_1^{IV}(x)}{5!}y^5 - \frac{f_0^{VI}(x)}{6!}y^6 - \cdots.$$

If we write

$$F_0(x) = |a_0| + |a_1|x + |a_2|x^2 + \cdots$$
$$F_1(x) = |b_0| + |b_1|x + |b_2|x^2 + \cdots \qquad (|x| < K),$$

we readily see that a necessary and sufficient condition for the convergence of S at a point (x, y) neither of whose coördinates is negative is the convergence of the two series

$$(5) \qquad F_0(x) + \frac{F_0''(x)}{2!}y^2 + \frac{F_0^{IV}(x)}{4!}y^4 + \cdots,$$

$$(6) \qquad F_1(x)y + \frac{F_1''(x)}{3!}y^3 + \frac{F_1^{IV}(x)}{5!}y^5 + \cdots.$$

* The regions of convergence of S and (4) are, in general, different. Wherever S converges, (4) converges, but the converse is not necessarily true.

Let us also write

$$\Phi(x) = \int_0^x F_1(x)\,dx.$$

Since $F_0(x)$ and $\Phi(x)$ are analytic in the complex x-plane throughout the circle $|x| < K$, we see by the Cauchy-Taylor theorem that the developments

$$F_0(x+y) = F_0(x) + F_0'(x)y + \frac{F_0''(x)}{2!}y^2 + \cdots,$$

$$\Phi(x+y) = \Phi(x) + F_1(x)y + \frac{F_1'(x)}{2!}y^2 + \cdots,$$

are valid when x and y are real and

$$0 \leqq x < K, \qquad 0 \leqq y < K - x.$$

Consequently, for these same values of (x, y) the series (5) and (6), and accordingly also the double series S, converge. From the well known property that a double power-series has a region of convergence which, except possibly for boundary points, is symmetrical with regard to the coördinate axes, we now infer that the series S converges throughout the square

$$|x| + |y| < K.$$

In order to see whether we have thus determined the complete region of convergence of S, let us first confine our attention to positive (not zero) values of x and y. Let (x_0, y_0) be any point whose coördinates are both positive and such that $x_0 + y_0 > K$. We wish to prove that S does not converge at (x_0, y_0). For this purpose we consider two cases:

(*a*) $x_0 < K$. We make use here of a theorem given by HADAMARD,* which says:

If a power-series in x with real non-negative coefficients has a radius of convergence $R > 0$, the analytic function represented by this series has a singular point at $x = R$.

Applying this theorem to the developments of $F_0(x)$ and $\Phi(x)$ about the point $x = 0$, we see that these functions have singular points at $x = K_0$ and $x = K_1$ respectively. Considering first the function F_0, we see that $F_0(x_0 + y)$ has a singular point at the point $y = K_0 - x_0$. Since $F_0(x_0 - y)$ is analytic at this point, the function

$$\tfrac{1}{2}\left[F_0(x_0 + y) + F_0(x_0 - y)\right]$$

has a singular point at $y = K_0 - x_0$. The development of this function according to positive integral powers of y cannot then converge when $y > K_0 - x_0$.

* *La Série de Taylor*, p. 20–21. The restriction there made to *positive* coefficients is not necessary as an examination of the proof shows.

This development is precisely what the series (5) becomes when $x = x_0$. Consequently (5), and hence also S, does not converge when $x = x_0$, $y > K_0 - x_0$.

Similar reasoning applied to the function

$$\tfrac{1}{2}[\Phi(x_0 + y) - \Phi(x_0 - y)]$$

shows that (6), and consequently S, does not converge when $x = x_0$, $y > K_1 - x_0$. Since y_0 satisfies at least one of the inequalities

$$y_0 > K_0 - x_0, \qquad y_0 > K_1 - x_0,$$

we thus see that S does not converge at (x_0, y_0).

(*b*) $x_0 \geqq K$. If S converged at (x_0, y_0), it would converge throughout the rectangle

$$0 < x < x_0, \qquad 0 < y < y_0.$$

Within this rectangle we could obviously choose a point (x_0', y_0') such that

$$0 < x_0' < K, \qquad 0 < y_0', \qquad x_0' + y_0' > K.$$

The convergence of S at this point is in contradiction with what we have proved under (*a*).

We have thus proved that if

$$x_0 > 0, \qquad y_0 > 0, \qquad x_0 + y_0 > K,$$

the series S does not converge at (x_0, y_0). From this we infer, on account of the symmetry of the region of convergence of a double power-series, that if neither x_0 nor y_0 is zero and

$$|x_0| + |y_0| > K,$$

the series S does not converge at (x_0, y_0). Thus we see that the complete region of convergence of S is the interior of the square

$$|x| + |y| < K$$

with possibly some (or all) points on its boundary, and possibly some (or all) points outside of it on the two coördinate axes. Since on the coördinate axes the double power-series reduces to a simple power-series, it is clear that the interval of convergence on each coördinate axis extends out the same distance from the origin in the two opposite directions, and that beyond the points thus reached we have divergence. At each one of the four points which terminate the intervals of convergence on the coördinate axes we may of course have convergence or divergence.

Before going on to other matters, we will note that the results so far obtained hold without change for the differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

practically no modification being required in the proofs. In this case the region of convergence of the double power-series is bounded by characteristics of the differential equation.

2. The series of homogeneous polynomials. Let us now rearrange the double series S into a simple series P whose terms are homogeneous polynomials in (x, y) of ascending degrees. It is clear that the terms of P are themselves solutions of Laplace's equation, and since it is well known, and easily proved, that every homogeneous polynomial of the nth degree in (x, y) which satisfies Laplace's equation may be written in the form

$$(A \cos n\theta + B \sin n\theta) r^n,$$

where

$$r = \sqrt{x^2 + y^2}, \qquad \tan \theta = \frac{y}{x},$$

it follows that the series P may be written in the form *

$$P \qquad u(x, y) = \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) r^n.$$

Since this series has been obtained from S by a rearrangement of the terms and an insertion of parentheses, we see that P converges wherever S converges. We shall presently find that it also converges elsewhere.

For this purpose we first establish the

Lemma. If when $r = r_0$ the series P converges for a continuous range of values of θ, however short, it converges absolutely for all values of θ when $r < r_0$, and if r_0' is any positive constant less than r_0, it converges uniformly in (x, y) when $r < r_0'$.

Proof. From the convergence of the trigonometric series P when $r = r_0$ for a continuous range of values of θ, we infer by a well known theorem of G. CANTOR † that the coefficients

$$A_n r_0^n, \qquad B_n r_0^n$$

approach zero as n becomes infinite. Let us denote by M the maximum of the absolute values of these coefficients. The terms of P are then readily seen to be in absolute value not greater than the corresponding terms of the series

$$\sum_{n=0}^{\infty} 2M \left(\frac{r}{r_0}\right)^n,$$

and since, when $r < r_0$, this series converges, it follows that P converges absolutely when $r < r_0$.

* The coefficients here may readily be determined to have the values

$$A_n = a_n, \qquad B_n = \frac{b_{n-1}}{n}.$$

† Mathematische Annalen, vol. 4 (1871), p. 139.

An obvious modification of this proof shows that P converges uniformly when $r < r_0'$.

From the lemma just proved we infer at once that throughout the circle $r < K$ the series P converges absolutely, and that throughout any smaller circle which has the origin as center it converges uniformly; * for if r_0 is any positive constant less than K, we can always find on the circle $r = r_0$ an arc lying within the square $|x| + |y| < K$, and on this arc the series S, and therefore also P, converge.

If the series P converged throughout a continuum lying outside of the circle $r < K$, we see, by an application of our lemma, that it would converge uniformly throughout a circle of radius $K' > K$. By a theorem of HARNACK according to which a series of harmonic functions which is uniformly convergent throughout a two dimensional continuum can be differentiated term by term, $\partial u/\partial y$ could then be obtained when $r < K'$ by differentiating P term by term. If in this series, and also in P itself, we let $y = 0$, we should get two power-series in x which are convergent when $|x| < K'$, and which, when $|x| < K$, have the values $f_1(x)$ and $f_0(x)$ respectively. They have therefore by hypothesis the radii of convergence K_1 and K_0 respectively, at least one of which is less than K'. Thus the assumption that P converges throughout a continuum outside of the circle $r = K$ leads to a contradiction.

If on each ray which radiates from the origin O we lay off a distance OQ, either finite or infinite, the totality of all the points on the segments OQ, excluding the points Q themselves, may be called a *star* with O as center. The points Q may be called the vertices of this star. It is then readily seen that the complete region of convergence of any series whose terms are homogeneous polynomials in (x, y) is a star with O as center together with perhaps some or all of its vertices. For on any particular ray (except on the rays $x = 0$) we may write $y = \lambda x$, where λ is a constant, and when this value of y is substituted, the series becomes a power-series in x. This reasoning in fact shows that the star in question is symmetrical with regard to the origin. Except possibly at the vertices of the star, the convergence will be absolute.

Turning now to the series P above considered, we see that all the rays of its star will reach out at least as far as the circumference of the circle $r = K$, so that this circle forms, so to speak, a solid nucleus of the star. Outside of it the rays do not completely fill any sector however narrow and short. Simple examples † show that there may be even an infinite number of these external rays

* These facts can readily be deduced without making use of CANTOR'S Theorem by reference to the values of A_n and B_n given in the first footnote in this section.

† For instance, the series

$$\sum_{n=1}^{\infty} r^{n!} \sin(n!\theta)$$

obviously converges when $r < 1$. By CANTOR'S Theorem it does not converge for any continu-

issuing from every arc, however short, of the circle; and that some or all of these rays may even extend to infinity.

A question of fundamental importance, at which we have not yet looked, is whether the series S and P represent harmonic functions throughout the whole continuum where they converge. This question is of course to be answered in the affirmative in the case of S, since a power series which converges throughout a continuum can always be differentiated there as often as we please, term by term. In the case of P it is also to be answered in the affirmative, since by HARNACK's Theorem a series of harmonic functions uniformly convergent throughout a two dimensional continuum represents a function harmonic throughout this continuum.

If S converges on rays which project from the extremities of the horizontal diagonal of the square $|x| + |y| < K$, the function represented by S need not be the analytic extension of the harmonic function within the square, since the function $f_1(x)$ may have a singular point at the point $x = +K$ or $x = -K$, and this function is equal to the value of $\partial u/\partial y$ on the horizontal diagonal of the square. Similar reasoning applies to the vertical diagonal. If, however, u admits of analytic extension along a two dimensional strip surrounding any one of these four rays, this analytic extension, so far as its values on the ray itself are concerned, will clearly be given by the series S if this series converges on this ray.

Since at points on the sides of the square, which are not vertices, u is harmonic and is represented by the series P, it is evident that if S converges at any such point it represents u there.

Precisely similar results hold with regard to the series P and the rays which project from the circle $r = K$, as we see by turning the coördinate axes.

Turning finally to another question, we note that if the harmonic function u is extended analytically so far as possible, and if R denotes the distance from the origin to the nearest singular point of u, it follows readily either by the use of Poisson's Integral (cf. PICARD, loc. cit.) or by an application of the Cauchy-Taylor development of the analytic function of which u is the real part, that u can be developed into a series of the form P which converges throughout the circle $r < R$. Since two developments of the form P for one and the same function are impossible,* it follows that P itself converges when $r < R$. It can

ous range of values of θ when $r = 1$. When θ is any rational multiple of π, all the terms in the series after a certain point are zero, and hence the series, for these values of θ everywhere dense, converges for all values of r.

* We use here the easily established theorem: If the series

$$\phi_0 + \phi_1 + \phi_2 + \cdots,$$

where ϕ_n is a homogeneous polynomial of degree n in (x, y), converges to the value zero at all points of a sector

$$\theta_1 \leqq \theta \leqq \theta_2, \qquad 0 \leqq r \leqq \rho,$$

obviously converge in no larger circle, as otherwise u would be harmonic throughout this larger circle. Consequently we have $K = R$.

3. Recapitulation of Results. We are now in a position to recapitulate our main results as follows:

If $u(x, y)$ is a function harmonic throughout the neighborhood of the point (x_0, y_0), and if, when this function is continued analytically,* the distance from (x_0, y_0) to the nearest singular point of u which lies in the same sheet of the Riemann's Surface generated by the analytic continuation in which (x_0, y_0) lies is K, then

(a) The Taylor's development S of $u(x, y)$ about (x_0, y_0) converges throughout the square

$$\Sigma \qquad |x - x_0| + |y - y_0| < K;$$

(b)† The development of $u(x, y)$ in a series P proceeding according to homogeneous polynomials of ascending degrees in $x - x_0$ and $y - y_0$ converges and represents u throughout the circle

$$\Pi \qquad \sqrt{(x - x_0)^2 + (y - y_0)^2} < K;$$

(c) The series S converges throughout no continuum which does not lie in Σ, and P converges throughout no continuum which does not lie in Π;

(d) The only points outside of Σ where S can converge are, first, points on the boundary of Σ, and at these it represents u (except at the vertices of Σ in case u is not defined there); and, secondly, points on the lines $x = x_0$ and $y = y_0$, at which it may or may not represent u.

(e) The complete region of convergence of P is a star having (x_0, y_0) as center, and perhaps some or all of its vertices. At the points of this region outside of Π the series P may or may not represent u. It will represent it at every point of the region to which a rectilinear analytic extension from (x_0, y_0) is possible.

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where $\theta_2 > \theta_1$, $\rho > 0$, then all the polynomials ϕ vanish identically. This is at once obvious if we write $y = \lambda x$ and thus express the series as a power-series in x whose coefficients are polynomials in λ.

* In the following statements u is supposed to be defined only at the points reached by this analytic continuation. Thus the region of definition of u is a continuum whose boundary points are the singular points of u.

† The fact here stated is, of course, well known, and is restated here only for the sake of completeness.